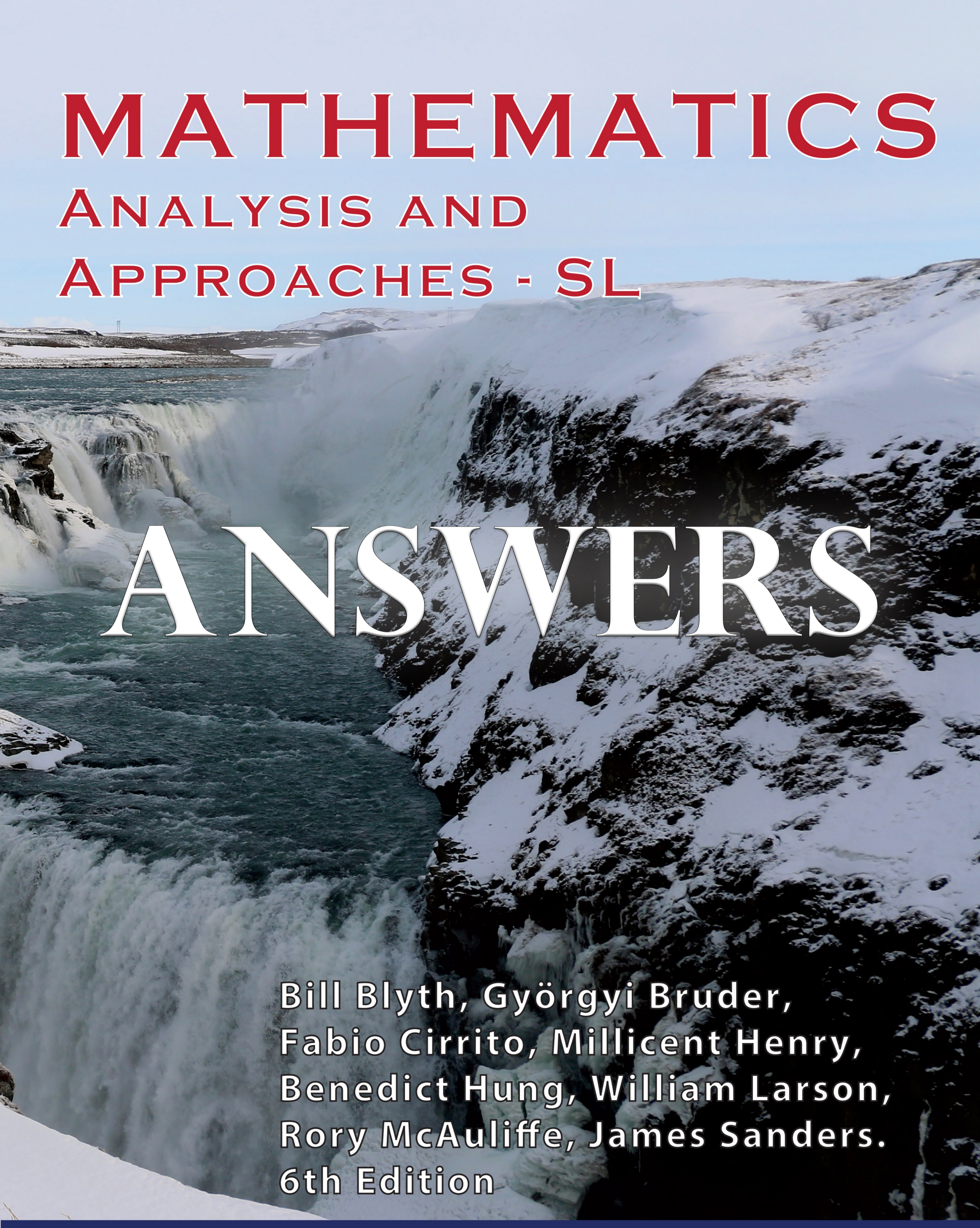
MATHEMATICS

Analysis and Approaches - SL



ANSWERS

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6th Edition

For use with the I.B. Diploma Programme

Exercise A.2.1

1 **a** $\frac{81}{2}$ **b** $\frac{10}{13}$ **c** 5000 **d** $\frac{30}{11}$

2 $23\frac{23}{99}$

3 6667 fish. [NB: $t_{43} < 1$. If we use $n = 43$ then ans is 6660 fish]; 20 000 fish.
Overfishing means that fewer fish are caught in the long run.

4 27

5 48,12,3 or 16,12,9

6 **a** $\frac{11}{30}$ **b** $\frac{37}{99}$ **c** $\frac{191}{90}$

7 128 cm

8 $\frac{121}{9}$

9 $2+\frac{4}{3}\sqrt{3}$

10 $\frac{1-(-t)^n}{1+t}$ $\frac{1}{1+t}$

11 $\frac{1-(-t^2)^n}{1+t^2}$ $\frac{1}{1+t^2}$

12 $^9/_5$ u^2.

Exercise A.3.1

1 **a** $\frac{27y^{15}}{8x^3}$ **b** $\frac{91}{216a^6}$ **c** $2^n + 2$ **d** $\frac{8x^{11}}{27y^2}$

e $\frac{3x^2y^2}{8}$ **f** $3^{n+1} + 3$ **g** $4^{n+1} - 4$

h $2(4^{n+1} - 4)$ **i** $\frac{1-b^6}{16b^4}$

2 **a** 64 **b** $\left(\frac{2}{3}\right)^x$ **c** 2^{2y+1} **d** $\frac{1}{b^{2x}}$

e $\left(\frac{y}{2}\right)^6$ **f** $\left(\frac{9}{2}\right)^{n+2}$

3 **a** $\frac{z^2}{xy}$ **b** 3^{7n-2} **c** 5^{n+1} **d** 9

e 2^{6n+1} **f** 2^{1-3n} **g** x^{2+4n-n^2}

h x^{3n^2+n+1} **i** 27

4 $\frac{y^{2m-2}}{x^m}$

5 **a** -81 **b** $-\frac{9x^8}{8y^4}$ **c** $y - x$

d $\frac{2x+1}{x+1}$ **e** -1 **f** $-b$

6 **a** $\frac{1}{x^2y^2}$ **b** $\frac{1}{x^4}$ **c** $-\frac{1}{x(x+h)}$

d $\frac{1}{x-1}$ **e** $\frac{1}{(x+1)(x-1)^5}$ **f** $\frac{1}{x^2}$

7 **a** $118 \times 5^{n-2}$ **b** 1 **c** $\frac{b^7}{a^4}$ **d** a^{mn}

e $\frac{p+q}{pq}$ **f** $\frac{2\sqrt{a}}{a-1}$ **g** $\frac{7}{8}$ **h** $a^{7/8}$

8 **a** $x^{11/12}$ **b** $2a^{3n-2}b^{2n-2}$ **c** 2^n

d $-\frac{7^{m-n}}{8}$ **e** $\frac{6 \times 5^n}{5^n + 5}$

Exercise A.3.2

1 **a** 2 **b** -2 **c** $\frac{2}{3}$ **d** 5 **e** 6

f -2.5 **g** 2 **h** 1.25 **i** $\frac{1}{3}$

2 **a** -6 **b** $-\frac{2}{3}$ **c** -3 **d** 1.5 **e** 0.25

f 0.25 **g** $-\frac{1}{8}$ **h** $-\frac{11}{4}$ **i** -1.25

Exercise A.3.3

1 **a** 2 **b** 2 **c** 5 **d** 3 **e** -3

f -2 **g** 0 **h** 0 **i** -1 **j** -2

k 0.5 **l** -2

2 **a** $\log_{10}10000 = 4$ **b** $\log_{10}0.001 = -3$
c $\log_{10}(x+1) = y$ **d** $\log_{10}p = 7$
e $\log_2(x-1) = y$ **f** $\log_2(y-2) = 4x$

3 **a** $2^9 = x$ **b** $b^x = y$ **c** $b^{ax} = t$
d $10^{x^2} = z$ **e** $10^{1-x} = y$ **f** $2^y = ax - b$

4 **a** 16 **b** 2 **c** 2 **d** 9 **e** $\sqrt[4]{2}$
f 125 **g** 4 **h** 9 **i** $\sqrt[3]{\frac{1}{3}}$ **j** 21 **k** 3
l 13

5 **a** 54.5982 **b** 1.3863 **c** 1.6487
d 7.3891 **e** 1.6487 **f** 0.3679
g 52.5982 **h** 4.7183 **i** 0.6065

Exercise A.3.4

1 **a** 5 **b** 2 **c** 2 **d** 1 **e** 2 **f** 1

2 **a** $\log a = \log b + \log c$ **b** $\log a = 2\log b + \log c$
c $\log a = -2\log c$ **d** $\log a = \log b + 0.5\log c$
e $\log a = 3\log b + 4\log c$ **f** $\log a = 2\log b - 0.5\log c$

3 **a** 0.18 **b** 0.045 **c** –0.09

4 **a** $x = yz$ **b** $y = x^2$ **c** $y = \frac{x+1}{x}$
d $x = 2^{y+1}$ **e** $y = \sqrt{x}$ **f** $y^2 = (x+1)^3$

5 **a** $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** $\frac{17}{15}$ **d** $\frac{3}{2}$ **e** $\frac{1}{3}$
f no real soln **g** 3,7 **h** $\frac{\sqrt{33}-1}{2}$ **i** 4
j $\sqrt{10}+3$ **k** $\frac{64}{63}$ **l** $\frac{2}{15}$

6 **a** $\log_3 2wx$ **b** $\log\frac{x}{47y}$ **c** $\log_a[x^2(x+1)^3]$
d $\log_a\left[\frac{(x^5)(x+1)^3}{\sqrt{2x-3}}\right]$ **e** $\log_{10}\left(\frac{y^2}{x}\right)$ **f** $\log_2\left(\frac{y}{x}\right)$

7 **a** 1 **b** –2 **c** 3 **d** 9 **e** 2 **f** 9

8 **a** 1,4 **b** $1, 3^{\pm\sqrt{3}}$ **c** $1, 4^{\sqrt[3]{4}}$ **d** $1, 5^{\pm\sqrt[4]{5}}$

9 **a** $\frac{\log 14}{\log 2} = 3.81$ **b** $\frac{\log 8}{\log 10} = 0.90$ **c** $\frac{\log 125}{\log 3} = 4.39$
d $\frac{1}{\log 2}\times\log\left(\frac{11}{3}\right) - 2 = -0.13$ **e** $\frac{\log 10 - \log 3}{4\log 3} = 0.27$
f 5.11 **g** $\frac{-\log 2}{2\log 10} = -0.15$

	h	7.37	**i**	0.93	**j**	no real solution			
	k	$\frac{\log 3}{\log 2} - 2 = -0.42$			**l**	$\frac{\log 1.5}{\log 3} = 0.37$			
10	**a**	0.5,4	**b**	3	**c**	−1,4	**d**	$10, 10^{10}$	**e** 5
	f	3							
11	**a**	$(4, \log_4 11)$	**b**	100,10	**c**	2,1			
12	**a**	$y = xz$	**b**	$y = x^3$	**c**	$x = e^{y-1}$			
13	**a**	$\frac{1}{e^4 - 1}$	**b**	$\frac{1}{3}$	**c**	$\frac{\sqrt{5}-1}{2}$	**d**	$\varnothing$	
14	**a**	$\ln 21 = 3.0445$	**b**	$\ln 10 = 2.3026$	**c**	$-\ln 7 = -1.9459$			
	d	$\ln 2 = 0.6931$	**e**	$\ln 3 = 1.0986$					
	f	$2\ln\left(\frac{14}{9}\right) = 0.8837$	**g**	$e^3 = 20.0855$					
	h	$\frac{1}{3}e^2 = 2.4630$	**i**	$\pm\sqrt{e^9} = \pm 90.0171$	**j**	$\varnothing$			
	k	$e^2 - 4 = 3.3891$	**l**	$\sqrt[3]{e^9} = 20.0855$					
15	**a**	$0, \ln 2$	**b**	$\ln 5$	**c**	$\ln 2, \ln 3$	**d**	0	
	e	$0, \ln 5$	**f**	$\ln 10$					
16	**a**	4.5222	**b**	0.2643	**c**	0,0.2619			
	d	−1,0.3219	**e**	−1.2925,0.6610	**f**	0,1.8928			
	g	0.25,2	**h**	1	**i**	121.5	**j**	2	

Exercise A.4.1

1 a $b^2 + 2bc + c^2$ b $a^3 + 3a^2g + 3ag^2 + g^3$

c $1 + 3y + 3y^2 + y^3$ d $16 + 32x + 24x^2 + 8x^3 + x^4$

e $8 + 24x + 24x^2 + 8x^3$ f $8x^3 - 48x^2 + 96x - 64$

g $16 + \frac{32}{7}x + \frac{24}{49}x^2 + \frac{8}{343}x^3 + \frac{1}{2401}x^4$ h $8x^3 - 60x^2 + 150x - 125$

i $27x^3 - 108x^2 + 144x - 64$ j $27x^3 - 243x^2 + 729x - 729$

k $8x^3 + 72x^2 + 216x + 216$ l $b^3 + 9b^2d + 27bd^2 + 27d^3$

m $81x^4 + 216x^3y + 216x^2y^2 + 96xy^3 + 16y^4$

n $x^5 + 15x^4y + 90x^3y^2 + 270x^2y^3 + 405xy^4 + 243y^5$

o $\frac{125}{p^3} + \frac{150}{p} + 60p + 8p^3$ p $\frac{16}{x^4} - \frac{32}{x} + 24x^2 - 8x^5 + x^8$

q $q^5 + \frac{10q^4}{p^3} + \frac{40q^3}{p^6} + \frac{80q^2}{p^9} + \frac{80q}{p^{12}} + \frac{32}{p^{15}}$

r $x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}$

Exercise A.4.2

1 a $160x^3$ b $21x^5y^2$ c $-448x^3$

d $-810x^4$ e $216p^4$

f $-20412p^2q^5$ g $-22680p$

2 a -1400000 b 6000 c 540

d -240 e 81648 f 40

3 1.0406 0.0004%

4 a $64x^6 + 960x^5 + 6000x^4 + 20000x^3 + 37500x^2 + 37500x + 15625$

b 19750 c 20.6 d 0.1%

5 19

6 $-\frac{63}{8}$

7 $\frac{231}{16}$

8 $-\frac{130}{27}$

9 -20

10 $a = \pm 3$

11 $n = 5$

12 $n = 9$

13 a 0 b -59

14 $a = 3, n = 8$

15 $a = \pm 2, b = \pm 1$

16. a $a = -3$ $n = 6$

b $a = {}^{1}/_{3}$ $n = 5$

c $a = \sqrt{2}$ $n = 4$

d $a = -\dfrac{\sqrt{2}}{6}$ $n = 4$

Exercise B.2.1

1 a i $f+g$: $[0,\infty[\mapsto\mathbb{R}$ where $(f+g)(x) = x^2+\sqrt{x}$ $[0,\infty[$

ii $f+g$: $]0,\infty[\mapsto\mathbb{R}$ where $(f+g)(x) = \frac{1}{x}+\ln(x)$ $[1,\infty[$

iii $f+g$: $[-3,-2]\cup[2,3]\mapsto\mathbb{R}$ where $(f+g)(x) = \sqrt{9-x^2}+\sqrt{x^2-4}$, $[\sqrt{5},\sqrt{10}]$

b i fg: $[0,\infty[\mapsto\mathbb{R}$ where $(fg)(x) = x^2\sqrt{x} = x^{5/2}$

ii fg: $]0,\infty[\mapsto\mathbb{R}$ where $(fg)(x) = \frac{\ln(x)}{x}$

iii fg: $[-3,-2]\cup[2,3]\mapsto\mathbb{R}$ where $(fg)(x) = \sqrt{(9-x^2)(x^2-4)}$

2 a i $f-g$: $]-\infty,\infty[\mapsto\mathbb{R}$ where $(f-g)(x) = 2e^x-1$ $]-1,\infty[$

ii $f-g$: $]-1,\infty[\mapsto\mathbb{R}$ where $(f-g)(x) = (x+1)-\sqrt{x+1}$ $]-0.25,\infty[$

iii $f-g$: $]-\infty,\infty[\mapsto\mathbb{R}$ where $(f-g)(x) = |x-2|-|x+2|$, $[-4,4]$

b i f/g: $\mathbb{R}\backslash\{0\}, \mapsto\mathbb{R}$ where $(f/g)(x) = \frac{e^x}{1-e^x}$

ii f/g: $]-1,\infty[\mapsto\mathbb{R}$ where $(f/g)(x) = \sqrt{x+1}$

iii f/g: $\mathbb{R}\backslash\{-2\}\mapsto\mathbb{R}$ where $(f/g)(x) = \left|\frac{x-2}{x+2}\right|$

3 a $fog(x) = x^3+1$, $gof(x) = (x+1)^3$ b $]-\infty,\infty[$, $]-\infty,\infty[$

ii a $fog(x) = x+1, x\geq 0$, $gof(x) = \sqrt{x^2+1}$ b $[1,\infty[$, $[1,\infty[$

iii a $fog(x) = x^2$, $gof(x) = (x+2)^2-2$ b $[0,\infty[$, $[-2,\infty[$

iv a $fog(x) = x, x\neq 0$, $gof(x) = x, x\neq 0$ b $\mathbb{R}\backslash\{0\}$, $\mathbb{R}\backslash\{0\}$

v a $fog(x) = x, x\geq 0$, $gof(x) = |x|$ b $[0,\infty[$, $[0,\infty[$

vi a $fog(x) = \frac{1}{x^2}-1, x\neq 0$, $gof(x)$ does not exist. b $]-1,\infty[$

vii a $fog(x) = x^2, x\neq 0$, $gof(x) = x^2, x\neq 0$ b $]0,\infty[$, $]0,\infty[$

viii a $fog(x) = |x|-4$, $gof(x) = |x-4|$ b $[-4,\infty[$, $[0,\infty[$

ix a $fog(x) = |x+2|^3-2$, $gof(x) = |x^3|$ b $[-2,\infty[$, $[0,\infty[$

x a $fog(x)$ does not exist, $gof(x) = (4-x), x\leq 4$ b $[0,\infty[$

xi a $fog(x) = \frac{x^2}{x^2+1}$, $gof(x) = \left(\frac{x}{x+1}\right)^2, x\neq -1$ b $[0,1[$, $[0,\infty[$

xii a $fog(x) = x^2+|x|+1$, $gof(x) = |x^2+x+1|$ b $[1,\infty[$, $[0.75,\infty[$

xiii a $fog(x) = 2^{x^2}$, $gof(x) = 2^{2x}$ b $[1, \infty[$, $]0, \infty[$

xiv a $fog(x)$ does not exist, $gof(x) = \frac{1}{x+1} - 1, x \neq -1$ b $\mathbb{R}\backslash\{-1\}$

xv a $fog(x)$ does not exist, $gof(x) = \frac{4}{x-1} + 1$ b $]1, \infty[$

xvi a $fog(x) = 4^{\sqrt{x}}, x \geq 0$, $gof(x) = 4^{0.5x}$ b $[1, \infty[$, $]0, \infty[$

4 a $fog(x) = 2x + 3, x \in \mathbb{R}$
b $gof(x) = 2x + 2, x \in \mathbb{R}$
c $fof(x) = 4x + 3, x \in \mathbb{R}$

5 $g(x) = x^2 + 1, x \in \mathbb{R}$

6 a $fog(x) = \frac{1}{x} + x + 1, x \in \mathbb{R}\backslash\{0\}$, $]-\infty,-1] \cup [3,\infty[$
b $gof(x)$ does not exist.

c $gog(x) = x + \frac{1}{x} + \frac{x}{x^2+1}, x \neq 0$, $]-\infty,-2.5] \cup [2.5,\infty[$

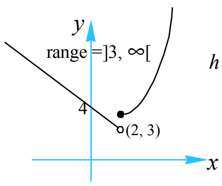
7 a 9 b 3

9 $g(x) = x^2 + 3$

10 $g(x) = \frac{1}{2}\sqrt{x^2 - 1} + 2$

11 a $x = \pm 1$ b $x = 1, -3$

12 a $\frac{1}{x}$ b $\frac{-x}{2x+1}$

13 

$hof(x) = \begin{cases} (x-1)^2 + 4, x \geq 2 \\ 5 - x, x < 2 \end{cases}$

14 a $r_f \subseteq d_g$ and $r_{gof} \subseteq d_h$ b $g(x) = 4(x+1)^2, x \in \mathbb{R}$

15 a $fog(x) = x, x \in]0,\infty[$ range = $]0,\infty[$

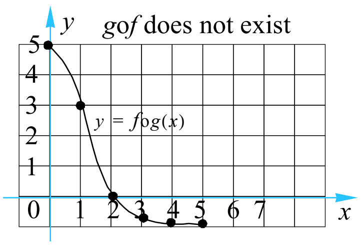
b $gof(x) = \frac{1}{2}(\ln(e^{2x-1}) + 1), x \in \mathbb{R}$ $(= x)$ range = $]-\infty,\infty[$

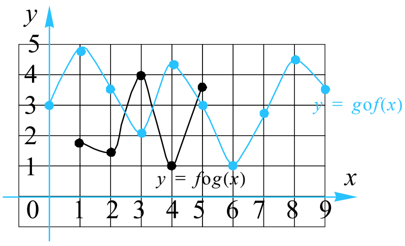
c $fof(x) = e^{2(e^{2x-1})-1}, x \in \mathbb{R}$ range = $]e^{-1}$,¥[

16 a hok does not exist. b $koh(x) = 4\log(4x-1) - 1, x > \frac{1}{4}$, $\mathbb{R}$

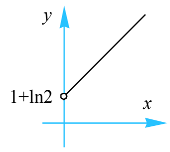
17 a $S = \mathbb{R}\backslash]-3,3[$; $T = \mathbb{R}$
b $T = \{x : |x| \geq 6, x = 0\}$; $S =]-\infty,-3] \cup [3,\infty[$

18





19 a Dom f =]0,∞[, ran f =]e,∞[, Dom g =]0,∞[, ran $g = \mathbb{R}$



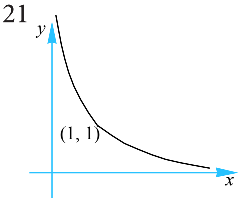
b fog does not exist: $r_g = \mathbb{R} \not\subset d_f$ =]0,∞[

gof exists as r_f =]e, ∞[$\subseteq d_g$ =]0,∞[

c gof:]0,∞[$\mapsto \mathbb{R}$, where $gof(x) = (x+1) + \ln 2$

20 $(fog)(x) = |x|, x \in \mathbb{R}$; range =[0, ∞[

21



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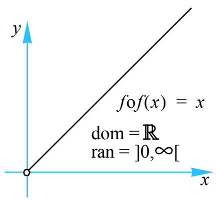
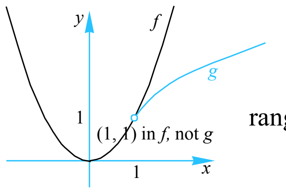


22



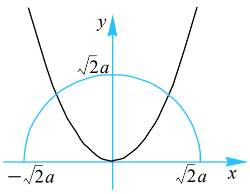
range =]1,∞[

b gof:]1,∞[$\mapsto \mathbb{R}$, where $gof(x) = x$

d fog^*:]1,∞[$\mapsto \mathbb{R}$, where $gof(x) = x$

$d_f = \mathbb{R} \setminus \left\{\frac{a}{c}\right\}, r_f = \mathbb{R} \setminus \left\{\frac{a}{c}\right\}, r_f \subseteq d_f, fof(x) = x$

24 a



b $d_{fog} = [-\sqrt{2a}, \sqrt{2a}], fog = 2a - \frac{x^2}{a}$

c $d_{gof} = [-2^{1/4}a, 2^{1/4}a], fog = \frac{1}{a}\sqrt{2a^4 - x^4}$,

$range = [0, \sqrt{2a}]$

Exercise B.2.2

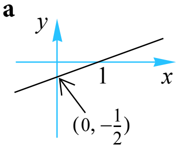
1 a $\frac{1}{2}(x-1), x \in \mathbb{R}$ b $\sqrt[3]{x}, x \in \mathbb{R}$

c $3(x+3), x \in \mathbb{R}$ d $\frac{5}{2}(x-2), x \in \mathbb{R}$

e $x^2 - 1, x > 0$ f $(x-1)^2, x \geq 1$

g $\frac{1}{x} - 1, x > 0$ h $\frac{1}{(x+1)^2}, x > -1$

2 a



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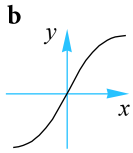

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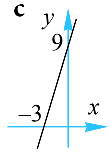

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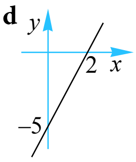

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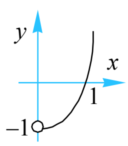

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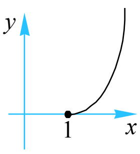

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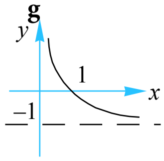

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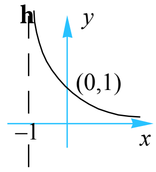
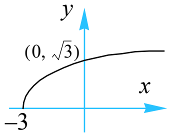
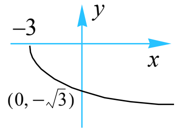


3 a $\sqrt{x+3}, x \geq -3$

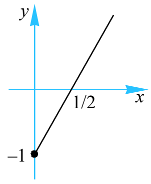


b $-\sqrt{x+3}, x \geq -3$

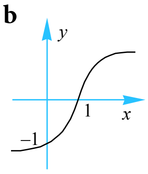


4 $\dfrac{\pm|x|}{\sqrt{1-x^2}}, -1 < x < 1$

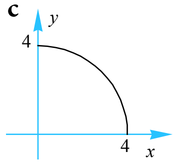
5 a



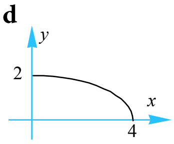
b



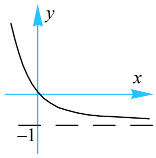
c



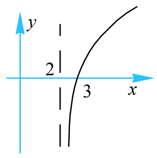
d



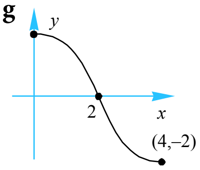
e



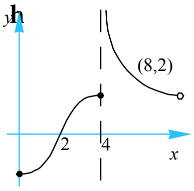
f



g



h



6

a $f^{-1}(x) = \log_3(x-1), x > 1$

b $f^{-1}(x) = \log_2(x+5), x > -5$

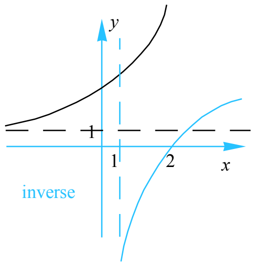
c $f^{-1}(x) = \frac{1}{2}(\log_3 x - 1), x > 0$

d $g^{-1}(x) = 1 + \log_{10}(3-x), x < 3$

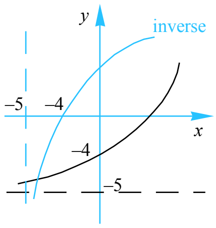
e $h^{-1}(x) = \log_3\left(1 + \frac{2}{x}\right), x \in \mathbb{R}\backslash[-2,0]$

f $g^{-1}(x) = \log_2\left(\frac{1}{x+1}\right), x > -1$

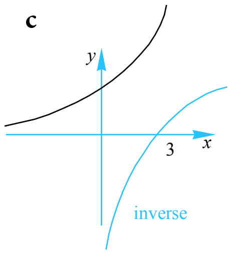
7 a



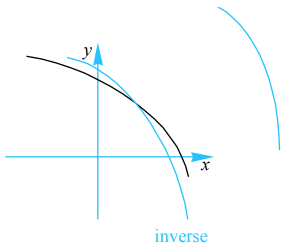
b



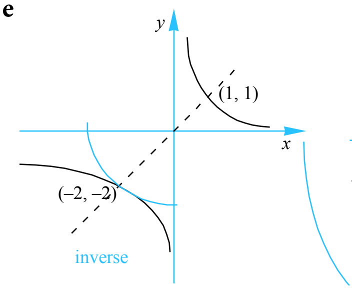
c



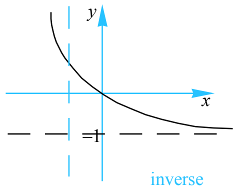
d



e



f



8 a $f^{-1}(x) = 2^x - 1, x \in \mathbb{R}$

b $f^{-1}(x) = \frac{1}{2} \cdot 10^x, x \in \mathbb{R}$

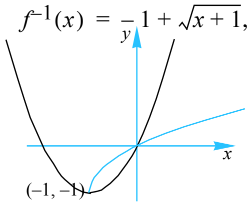
c $f^{-1}(x) = 2^{1-x}, x \in \mathbb{R}$

d $f^{-1}(x) = 3^{x+1} + 1, x \in \mathbb{R}$

e $f^{-1}(x) = 5^{x/2} + 5, x \in \mathbb{R}$

f $f^{-1}(x) = 1 - 10^{3(2-x)}, x \in \mathbb{R}$

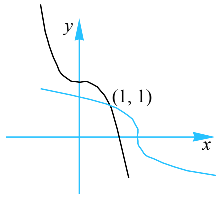
9 $f^{-1}(x) = -1 + \sqrt{x+1}, x > -1$



dom = [–1, ∞[, ran = [–1, ∞[

10 a $f^{-1}(x) = a - x$ **b** $f^{-1}(x) = \dfrac{2}{x-a} + a$ **c** $f^{-1}(x) = \sqrt{a^2 - x^2}$

11 $f^{-1}(x) = \sqrt[3]{2-x}$



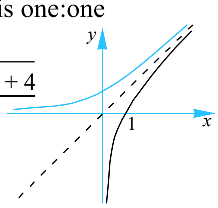
12 [2, ∞[

13 $\mathbb{R}^+ \backslash \{1.5\}$

14 a Inverse exists as *f* is one:one

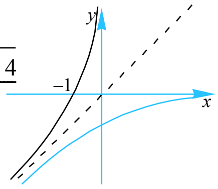
b Case 1: $S =]0, \infty[$

$g^{-1}(x) = \dfrac{x + \sqrt{x^2+4}}{2}$

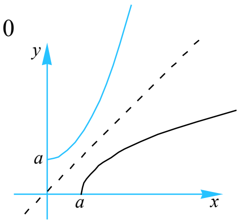


Case 2: $S =]-\infty, 0[$

$g^{-1}(x) = \dfrac{x - \sqrt{x^2+4}}{2}$

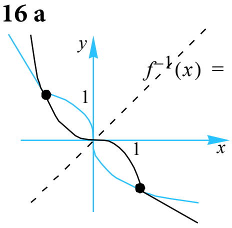


15 $f^{-1}(x) = a(x^2+1), x \ge 0$



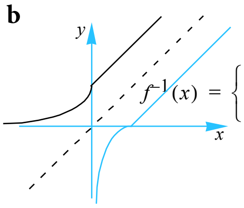
$\{x: f(x) = f^{-1}(x)\} = \varnothing$

16 a



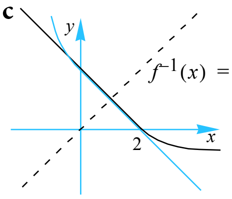
$f^{-1}(x) = \begin{cases} -(2x+1), & x < -1 \\ -\sqrt[3]{x}, & x \ge -1 \end{cases}$

b



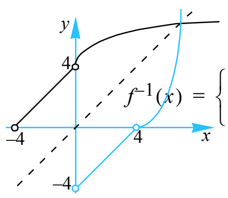
$f^{-1}(x) = \begin{cases} \ln(x) - 1, & 0 < x \le e \\ x - e, & x > e \end{cases}$

c



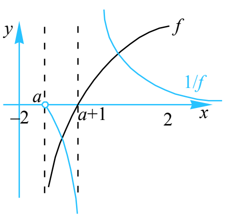
$f^{-1}(x) = \begin{cases} 1 + e^{-x}, & x < 0 \\ 2 - x, & x \ge 0 \end{cases}$

d

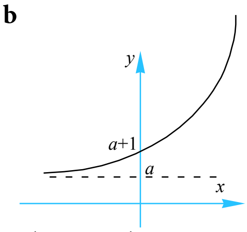


$f^{-1}(x) = \begin{cases} (x-4)^2, & x > 4 \\ x - 4, & 0 < x < 4 \end{cases}$

17 a



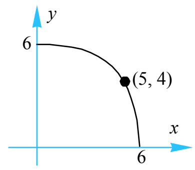
b



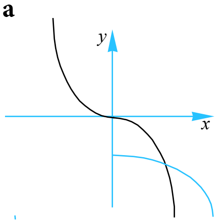
$f^{-1}: \mathbb{R} \mapsto \mathbb{R}, f^{-1}(x) = a + e^{ax}$

18 $g \circ f$ exists as $r_f \subseteq d_g$.

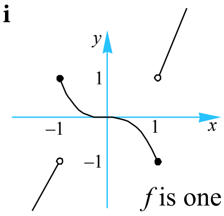
It is one:one so the inverse exists:



19 a



b i

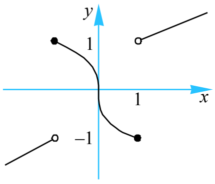


f is one:one

ii

$$f(x) = \begin{cases} \frac{1}{2}(x-1) & x < -1 \\ -\sqrt[3]{x} & -1 \le x \le 1 \\ \frac{1}{2}(x+1) & x > 1 \end{cases}$$

iii



iv $\{-1, 0, 1\}$

20 a i $tom(x) = e^{\sqrt{x}}, x \ge 0$ ii $mot(x) = \sqrt{e^x}, x \in \mathbb{R}$

b i $(tom)^{-1}(x) = (\ln(x))^2, x > 1$ ii $(mot)^{-1}(x) = \ln x^2, x > 0$

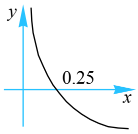
c i & ii neither exist

d Adjusting domains so that the functions in part c exist, we have:

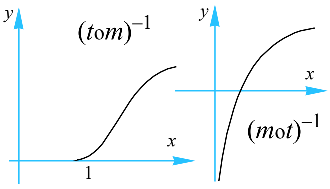
$t^{-1} o m^{-1}(x) = (mot)^{-1}(x)$ and $m^{-1} o t^{-1}(x) = (tom)^{-1}(x)$

e Yes as rules of composition OK.

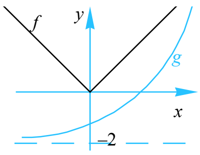
21 a 1 b



0.206



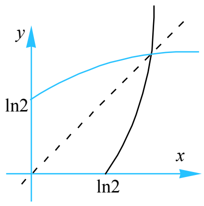
22 a



b $f \circ g$ exists but is not one:one

c i B = [ln2, ∞[

ii $(f \circ g)^{-1}: [0, \infty[\mapsto \mathbb{R}$ where, $(f \circ g)^{-1}(x) = \ln(x+2)$ iii

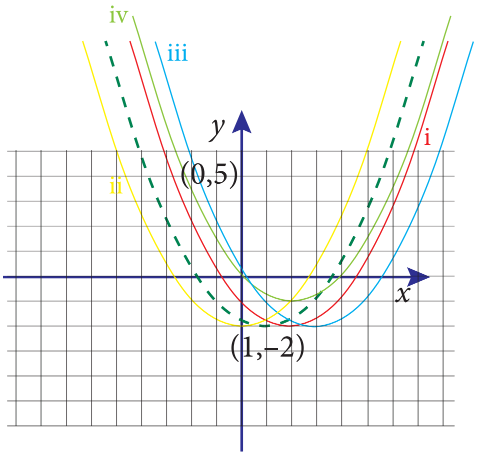


Exercise B.3.1

1. 1.853
2. 1.484
3. 1.379
4. 1.803
5. 1.618 (or 0)
6. 0.4429
7. 2.151
8. 1.400
9. 360
10. 9 - surprised? You probably had to use guess & check to do this as not many calculators will graph the factorial function.

Exercise B.3.2

1. a

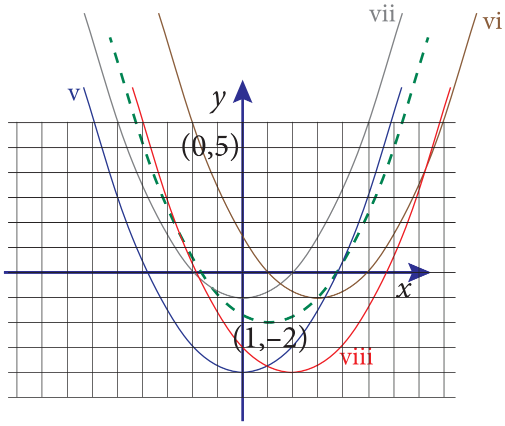


$y = f(x-1)$

$y = f(x+1)$

$y = f(x-2)$

$y = f(x-1)+1$



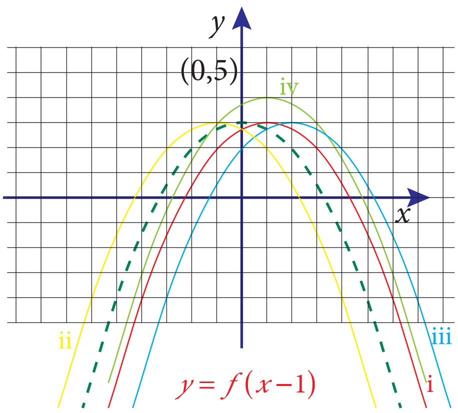
$y = f(x+1)-2$

$y = f(x-2)+1$

$y = f(x+1)+1$

$y = f(x-1)-2$

b

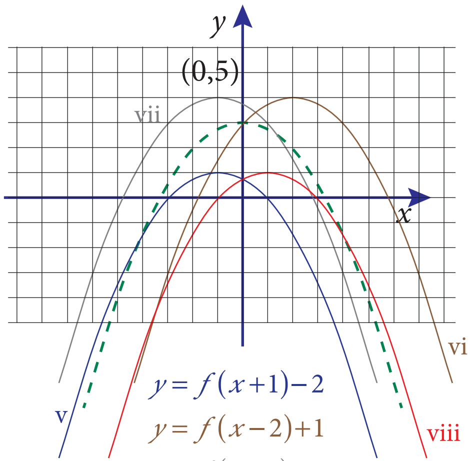


$y = f(x-1)$

$y = f(x+1)$

$y = f(x-2)$

$y = f(x-1)+1$



$y = f(x+1)-2$

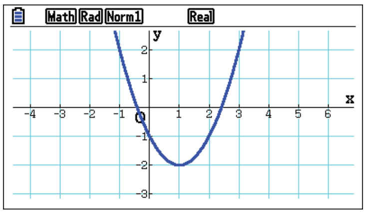
$y = f(x-2)+1$

$y = f(x+1)+1$

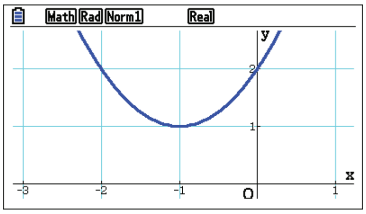
$y = f(x-1)-2$

2.

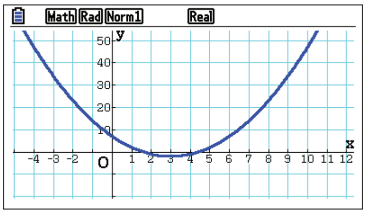
a



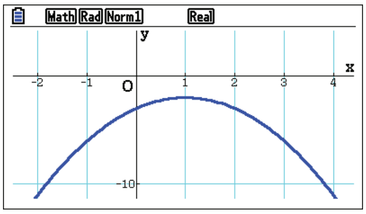
b



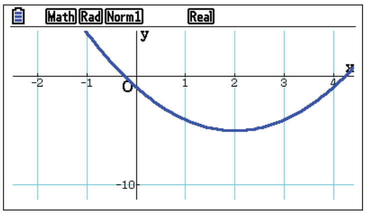
c



d



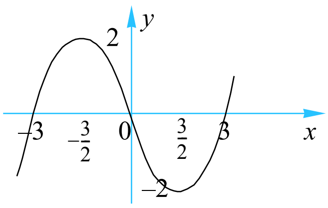
e



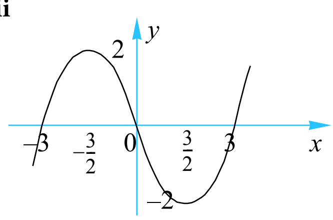
3. a $x = -2$ b $x = 4$ c $x = -1$ d $x = 12$

Exercise B.3.3

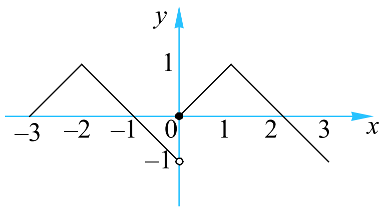
1 a i



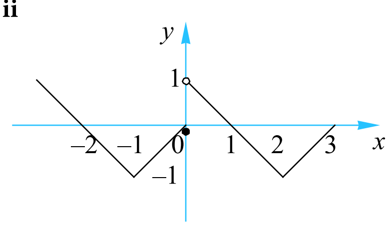
ii



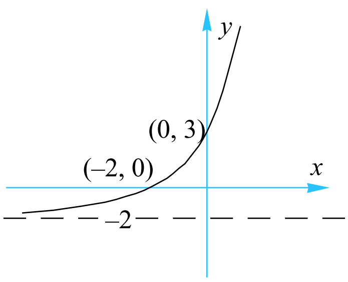
b i



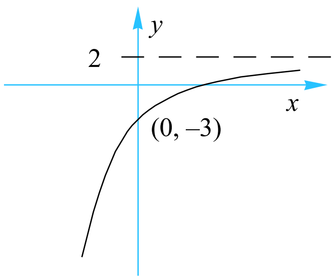
ii



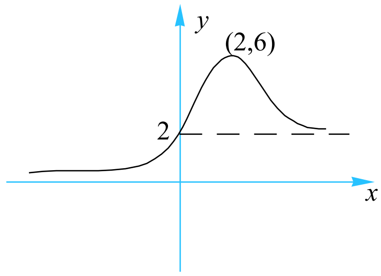
c i



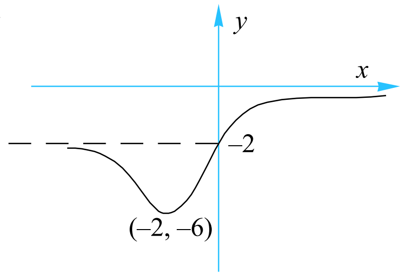
ii



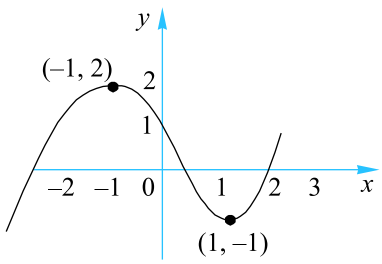
d i



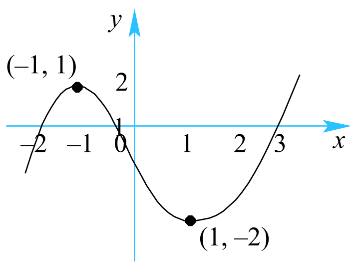
ii



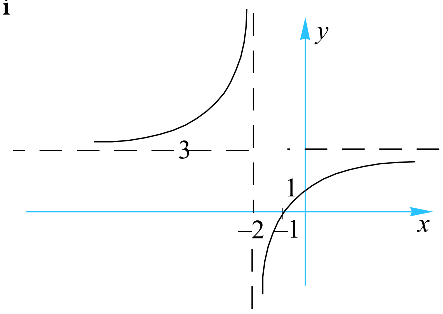
e i



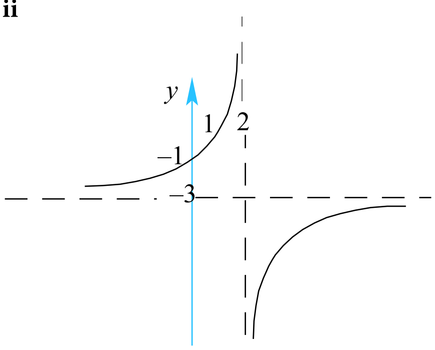
ii



f i



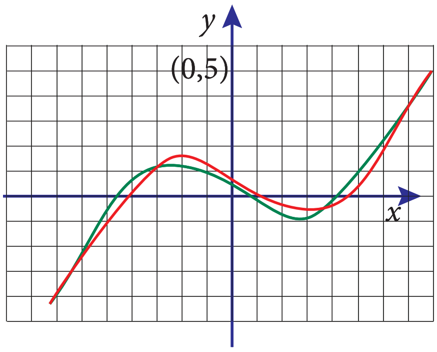
ii



2 a $y = -f(x)$ b $y = f(-x)$ c $y = f(x + 1)$

d $y = f(2x)$ e $y = 2f(x)$

3.



4. $f(x)$ --> $f(2x)$

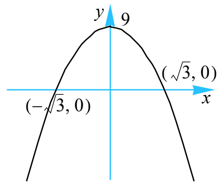
5. The temperatures must be scaled using $F = 1.8C + 32$

6. $f(x)$ --> $-2\,f(x + 2) + 4$

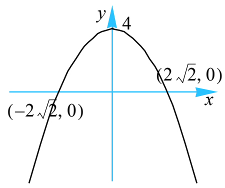
7. Adjusted reading = 1.1 × raw reading – 5

Exercise B.3.4

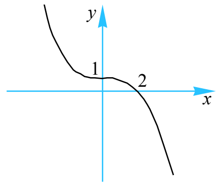
1 a



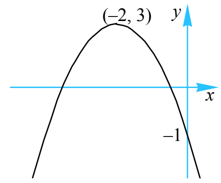
b



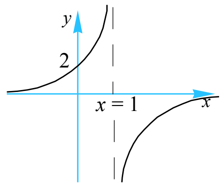
c



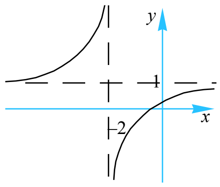
d



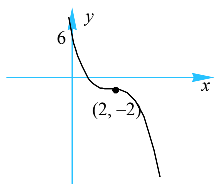
e



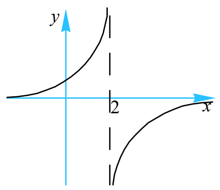
f



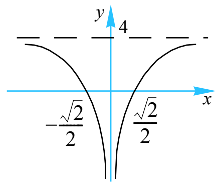
g



h



i



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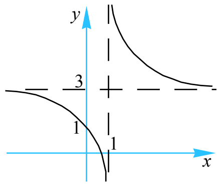

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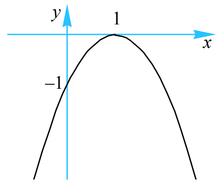
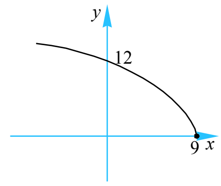
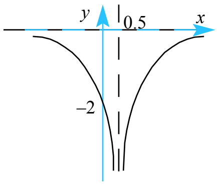


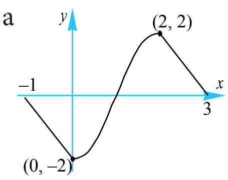
l



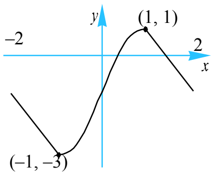
m



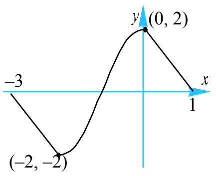
2 a



b



c



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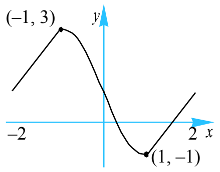
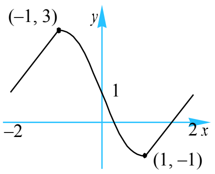
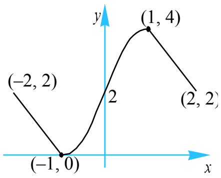


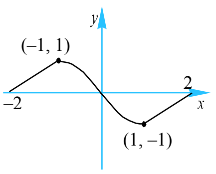
e



f



g



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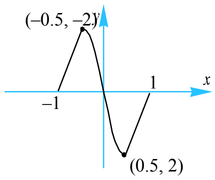
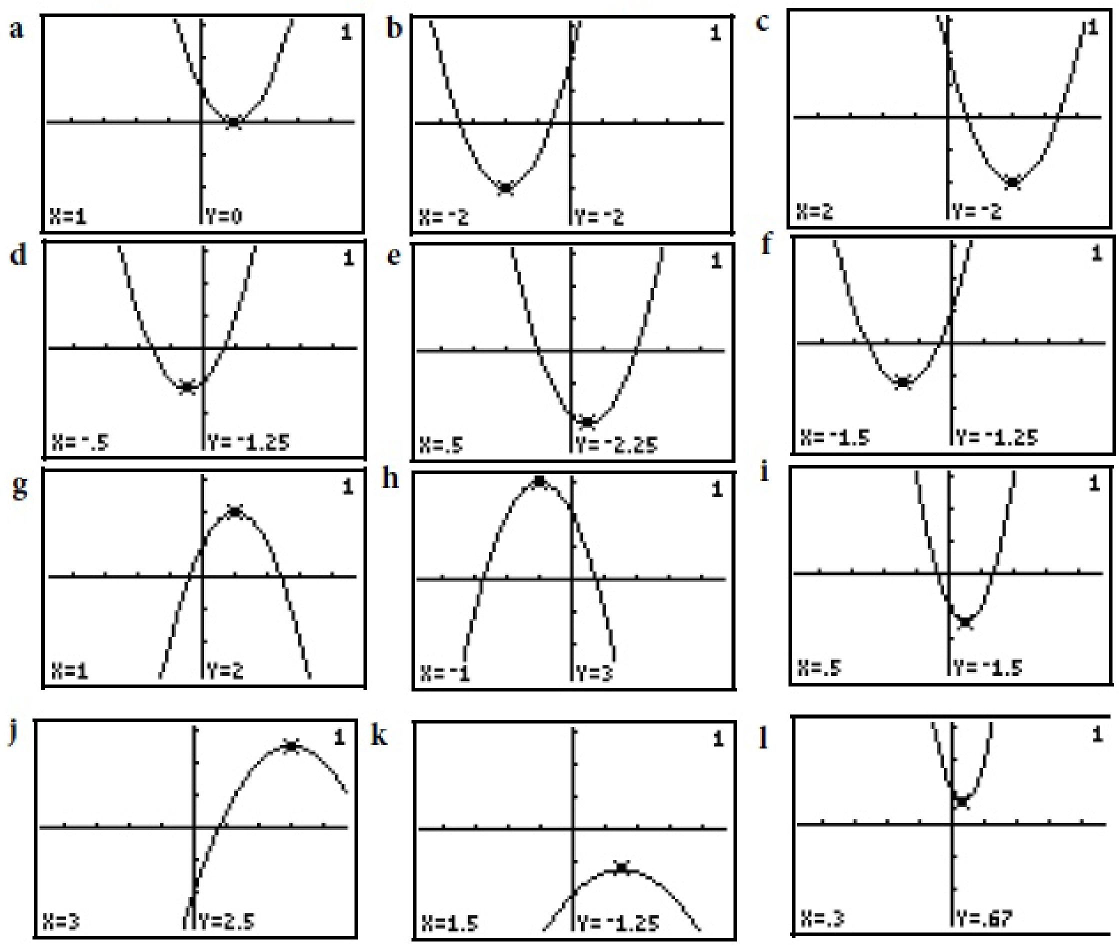
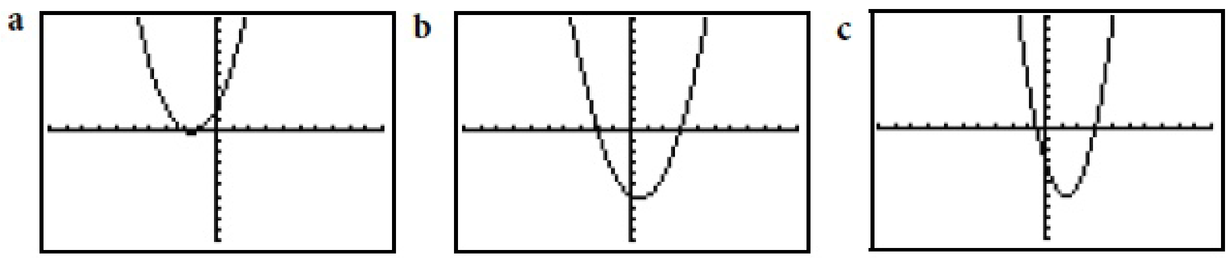


Exercise B.4.1

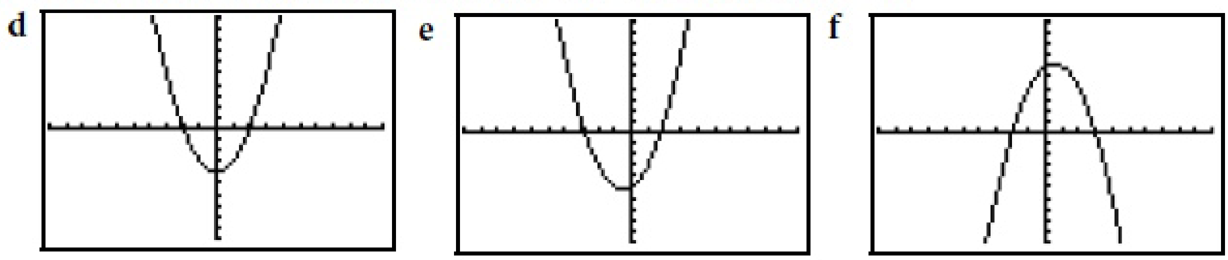
1.



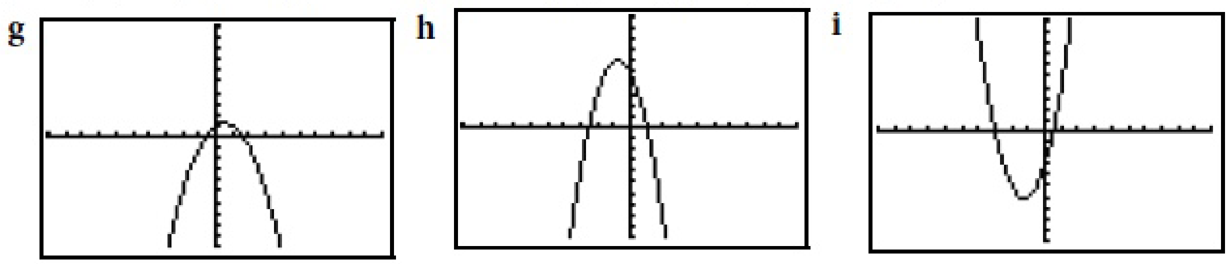
2.



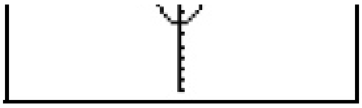
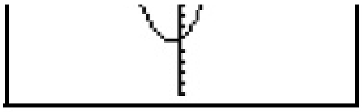
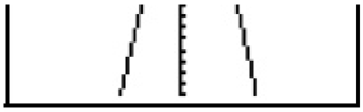
(–2, 0), (–1, 0), (0, 2)(–2, 0), (3, 0), (0, –6)(–0.5, 0), (3, 0), (0, –3)



(–2, 0), (2, 0), (0, –4)(–2.79, 0), (1.79, 0), (0, –5)(–2, 0), (3, 0), (0, 6)

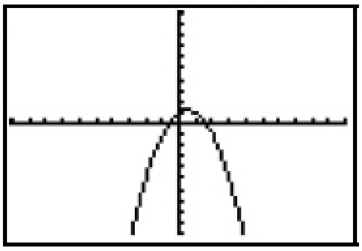


(–0.62, 0), (1.62, 0), (0, 1)(–2.5, 0), (1, 0), (0, 5)(–3, 0), (0.5, 0), (0, –3)

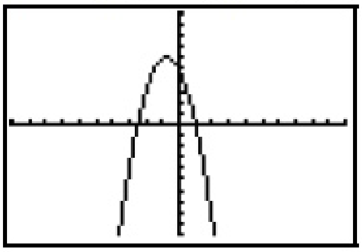




(−2, 0), (2, 0), (0, −4)(−2.79, 0), (1.79, 0), (0, −5)(−2, 0), (3, 0), (0, 6)

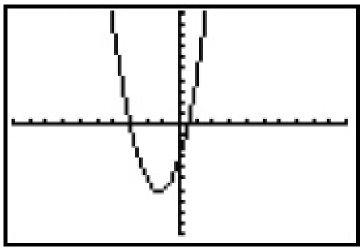
g



h

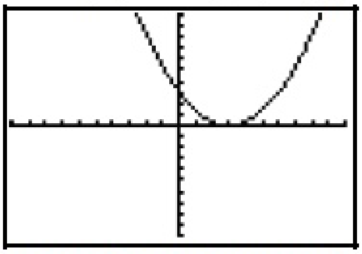


i

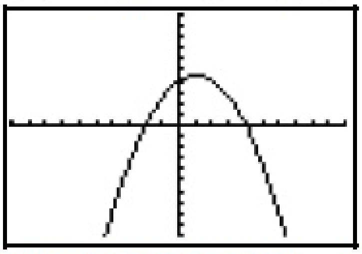


(−0.62, 0), (1.62, 0), (0, 1)(−2.5, 0), (1, 0), (0, 5)(−3, 0), (0.5, 0), (0, −3)

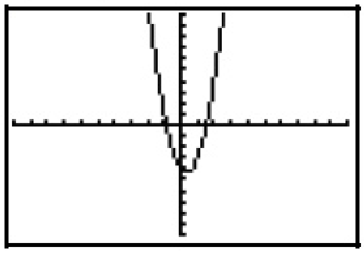
j



k

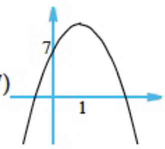


l



(3, 0), (0, 3)(−2, 0), (4, 0), (0, 4)(−0.87, 0), (1.54, 0), (0, −4)

4 **a** $x = 1$ **b** $(1, 9)$ **c i** $\left(\frac{2 \pm 3\sqrt{2}}{2}, 0\right)$ **ii** $(0, 7)$



5 **a** $k = \frac{9}{4}$ **b** $k < \frac{9}{4}$ **c** $k > \frac{9}{4}$

6 **a** $k = \frac{25}{8}$ **b** $k < \frac{25}{8}$ **c** $k > \frac{25}{8}$

7 **a** $k = \pm 1$ **b** $-1 < k < 1$ **c** $k < -1 \cup k > 1$

8 **a** $y = \frac{5}{12}(x-2)(x-6)$ **b** $y = -\frac{3}{8}(x+4)^2$

c $y = \frac{3}{4}(x-2)^2 + 1$ **d** $y = 3x^2 - 6x + 7$

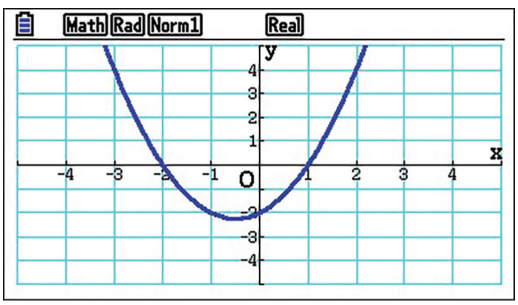
9 **a** $y = -\frac{2}{5}x(x-6)$ **b** $y = \frac{3}{4}(x-3)^2$

c $y = \frac{7}{9}(x+2)^2 + 3$ **d** $y = -\frac{7}{3}x^2 - 2x + \frac{40}{3}$

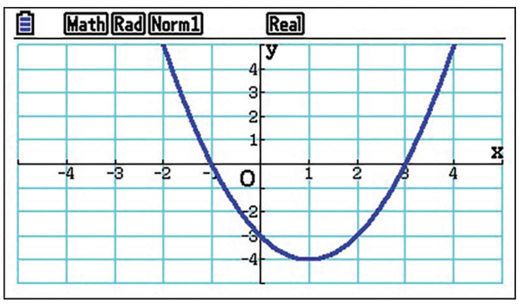
Exercise B.4.2

1.

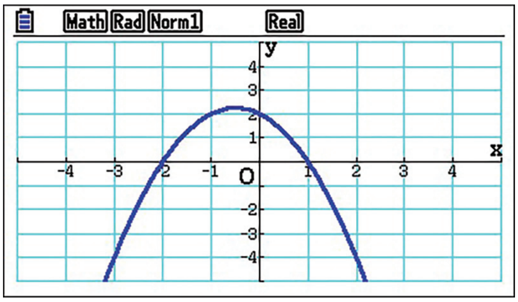
a



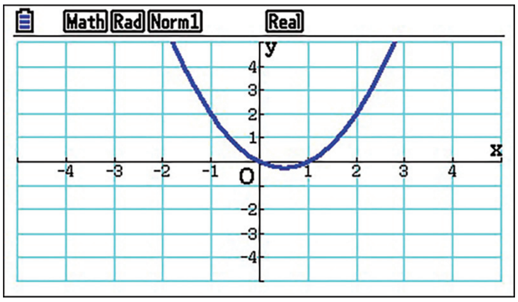
b



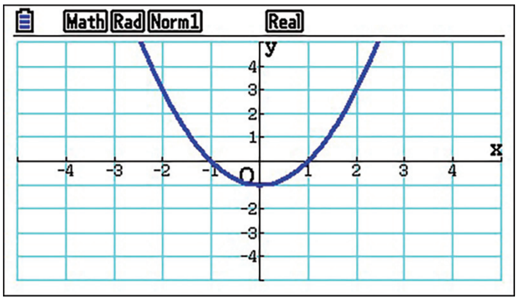
c



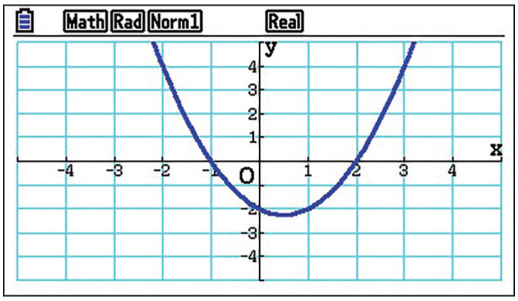
d



e

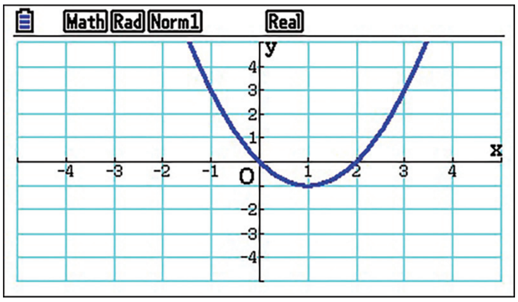


f

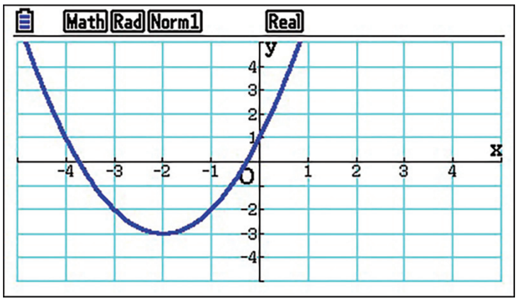


2.

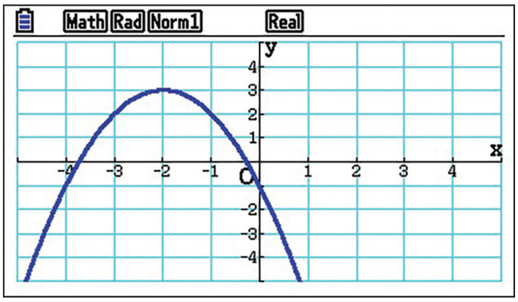
a



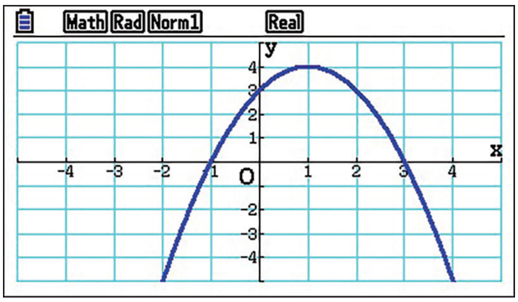
b



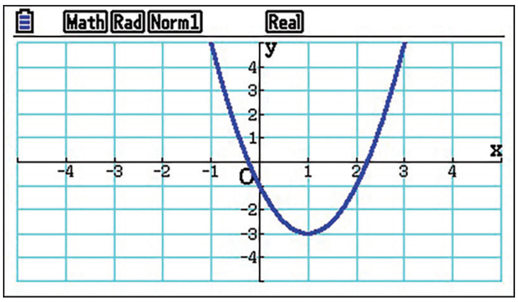
c



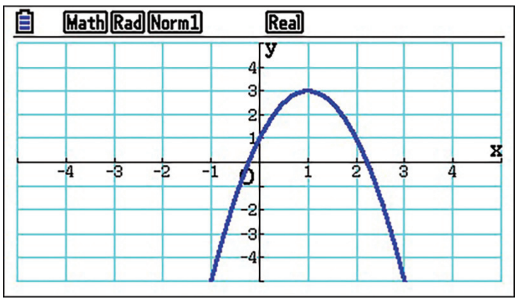
d



e

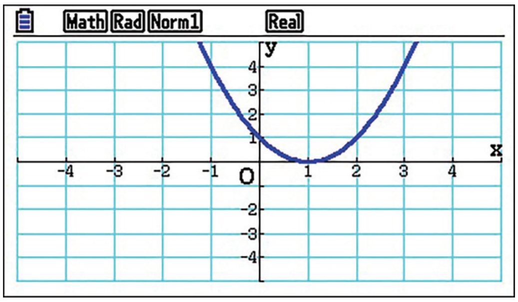


f

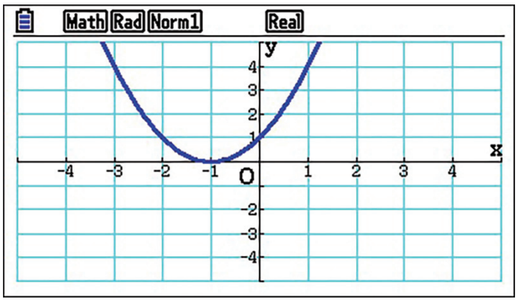


3.

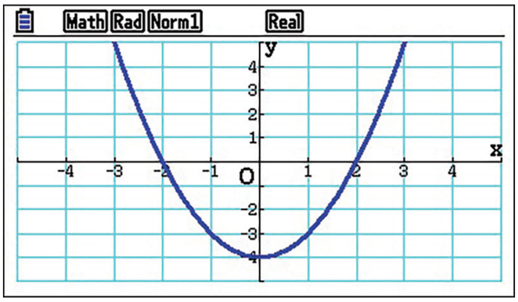
a



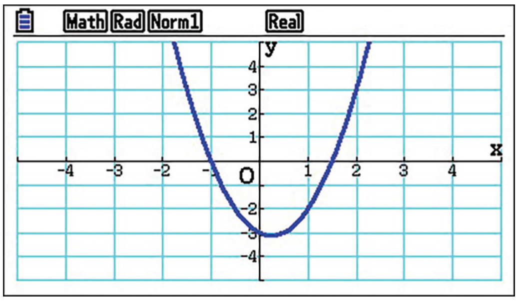
b



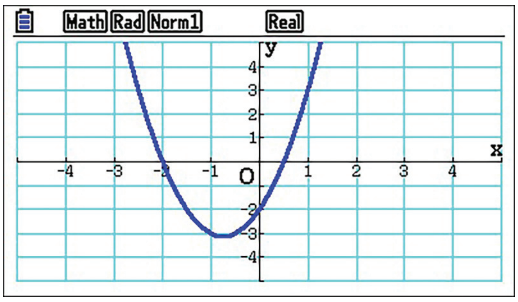
c



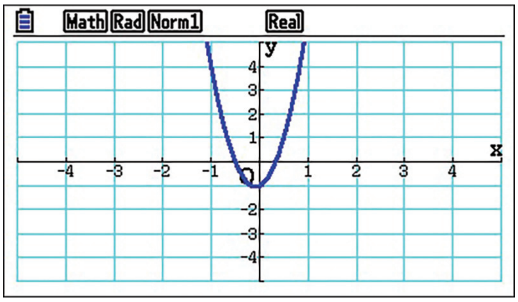
d



e

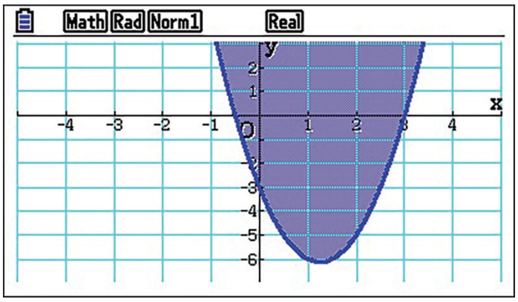


f

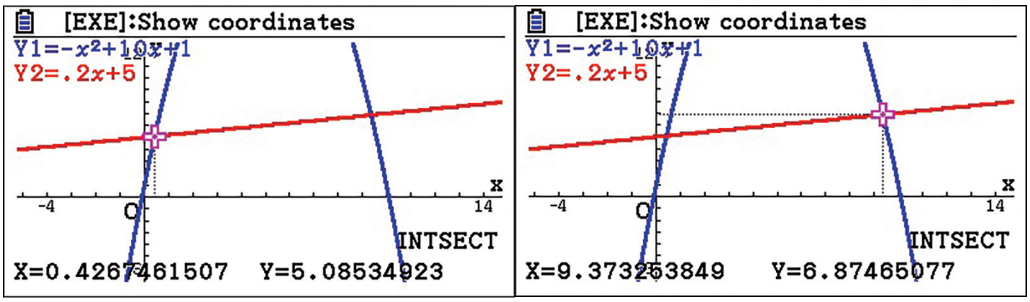


4. $y = 2(x-1)^2 = 4x^2 - 8x + 4$

5. $y = -x^2 + 4x - 3$

6. 

$-½ < x < 3$

7. 

Profitable from 0.43 to 9.37 years.

8. Yes - $h(4.5) = 0.225$ (22.5 metres) so the projectile passes over the deck.

Exercise B.4.3

1.	a	–5	b	4, 6	c	–3, 0	d	1, 3
	e	–6, 3	f	$-2, {}^5/_3$	g	2	h	–3, 6
	i	–6, 1	j	$0, {}^3/_2$				
2.	a	–1	b	–7, 5	c	$-{}^2/_5, 3$	d	–2, 1
	e	–3, 1	f	4, 5				
3.	a	$-1\pm\sqrt{6}$	b	$3\pm\sqrt{5}$	c	$1\pm\sqrt{5}$	d	$\frac{-1\pm\sqrt{33}}{8}$
	e	$\frac{9\pm\sqrt{73}}{4}$	f	$\frac{1\pm\sqrt{85}}{6}$				
4.	a	$\frac{3\pm\sqrt{37}}{2}$	b	$\frac{5\pm\sqrt{33}}{2}$	c	$\frac{3\pm\sqrt{33}}{2}$	d	$\frac{7\pm\sqrt{57}}{2}$
	e	$\frac{-7\pm\sqrt{65}}{2}$	f	–4, 2	g	$-1\pm2\sqrt{2}$	h	$\frac{-5\pm\sqrt{53}}{2}$
	i	$\frac{3\pm\sqrt{37}}{2}$	j	no real solutions			k	$4\pm\sqrt{7}$
	l	no real solutions			m	$\frac{2\pm\sqrt{13}}{2}$	n	$\frac{3\pm2\sqrt{11}}{5}$
	o	$\frac{6\pm\sqrt{31}}{5}$	p	$\frac{6\pm\sqrt{29}}{7}$				
5.	a	$-2 < p < 2$	b	$p = \pm2$	c	$p < -2$ or $p > 2$		
6.	a	$m = 1$	b	$m < 1$	c	$m > 1$		
7.	a	$\pm2\sqrt{2}$	b	$]-\infty,-2\sqrt{2}[\cup]2\sqrt{2},\infty[$	c	$]-2\sqrt{2},2\sqrt{2}[$		
8.	a	$k=\pm6\sqrt{2}$	b	$]-\infty,-6\sqrt{2}[\cup]6\sqrt{2},\infty[$	c	$]-6\sqrt{2},6\sqrt{2}[$		
10.	4							

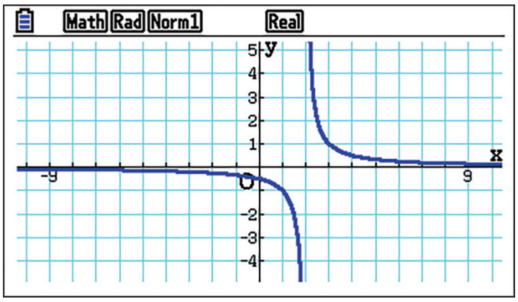
12.	i	$1\le x\le 3$	ii	$x\le -1$ or $x\ge 2$	iii	$-1-\sqrt{5}<x<-1+\sqrt{5}$
	iv	$\frac{-7-\sqrt{61}}{2}\le x\le\frac{-7+\sqrt{61}}{2}$	v	no solutions	vi	$x = -1$
	vii	$\frac{7-\sqrt{37}}{6}\le x\le\frac{7+\sqrt{37}}{6}$	viii	$\frac{-3-\sqrt{13}}{2}\le x\le\frac{-3+\sqrt{13}}{2}$		
	ix	no solutions	x	$\frac{1-\sqrt{5}}{2}\le x\le\frac{1+\sqrt{5}}{2}$ or $x < -1$		

13. 60%

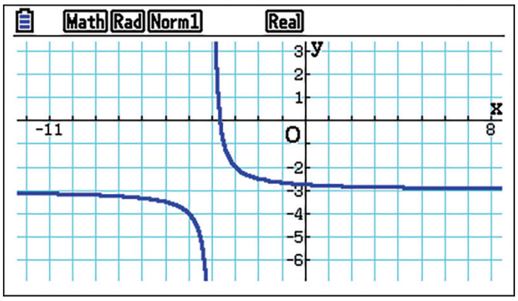
14. a $h = 6 + 2.8x .- 0.1x^2, 0 \le x \le 24$ b (6.52, 20) to (21.5, 20) ie. about 15 m.

Exercise B.4.4

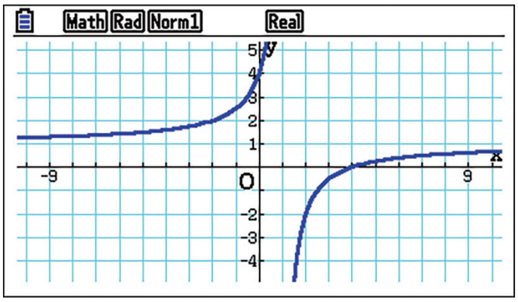
1. i



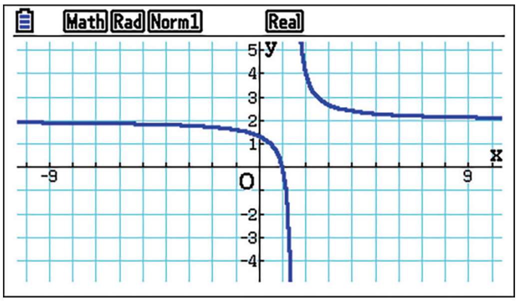
ii



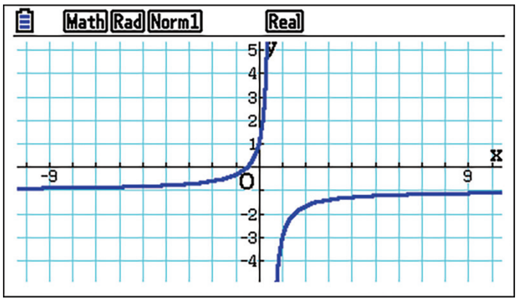
iii



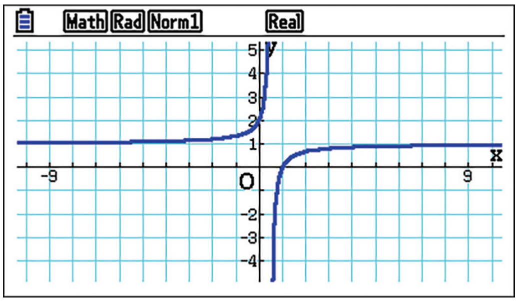
iv



v



vi



2. $y = \dfrac{1}{x-2} - 1, x \neq 2$

3. $y = \dfrac{2}{x-1} + 1, x \neq 1$

4. $y = 1 - x, y = x - 5$

5. $y = \pm x$

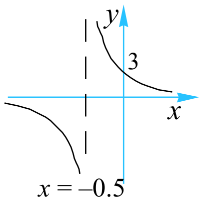
6. $P = \dfrac{470}{V}$

Exercise B.4.5

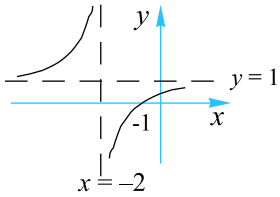
1 **a** $y = 2, x = -1$ **b** $y = 1, x = -\frac{1}{3}$ **c** $y = \frac{1}{2}, x = -\frac{1}{4}$

d $y = -1, x = -3$ **e** $y = 3, x = 0$ **f** $y = 5, x = 2$

3 a



b



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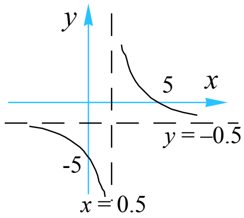

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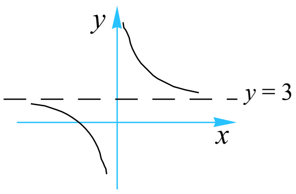

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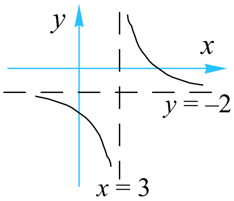

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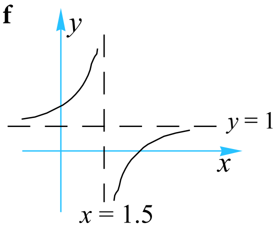
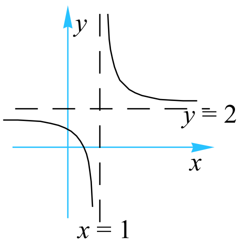
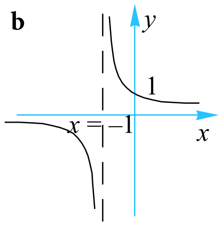


4 $a = 2, c = 4$

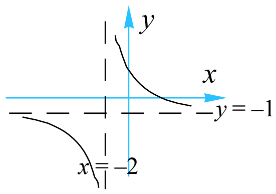
5 a



b

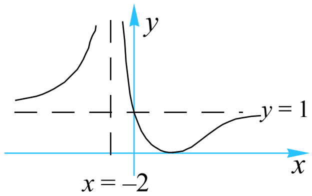


6 **a i** $(0, 1), (2, 0)$ **ii** $y = -1, x = -2$ **iii**

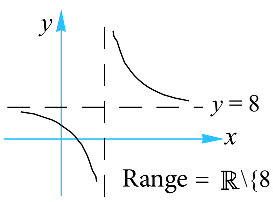


iv $d = \mathbb{R} \setminus \{-2\}$

b $f^{-1}: \mathbb{R}\setminus\{-1\} \mapsto \mathbb{R}$, where $f^{-1}(x) = \frac{2(1-x)}{(1+x)}$ **c**

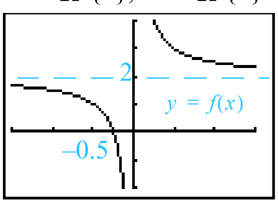


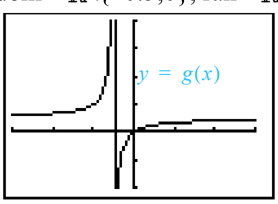
7 **a** $y = 8, x = 3$ **b**



Range = $\mathbb{R} \setminus \{8\}$

8 dom = $\mathbb{R} \setminus \{0\}$, ran = $\mathbb{R} \setminus \{2\}$ dom = $\mathbb{R} \setminus \{-0.5, 0\}$, ran = $\mathbb{R} \setminus \{0.5\}$





9 Asymptotes: **a** $y = 2x, x = 0$ **b** $y = \frac{1}{2}x, x = 0$ **c** $y = -x, x = 0$ **d** $y = x, x = 0$

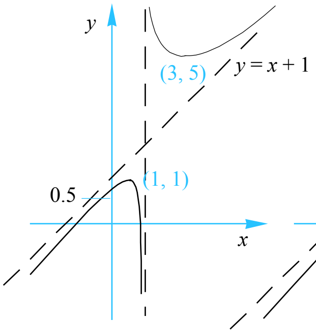
10 Asymptotes: **a** $y = x^2, x = 0$ **b** $y = x^2, x = 0$ **c** $y = x, x = 0$ **d** $y = x^3, x = 0$

11 Asymptotes: **a** $y = x+3, x = 0$ **b** $y = -x+2, x = 0$ **c** $y = 2x-2, x = 0$

d $y = x+2, x = 0$

12 **a i** (0, 4), (2, 0) **ii** $y = 3-x, x = 1$

13 **a**



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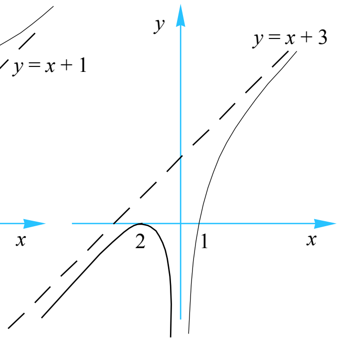
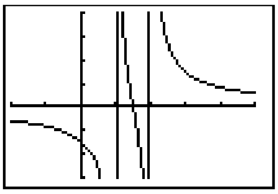


14 a



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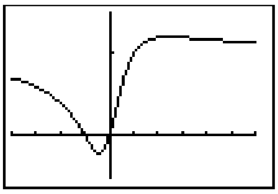

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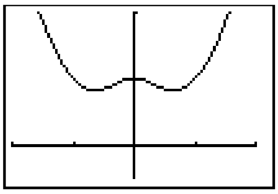
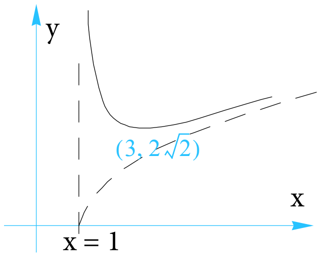
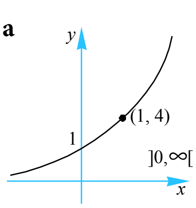


15

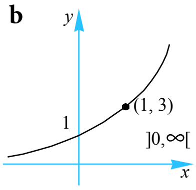


Exercise B.4.6

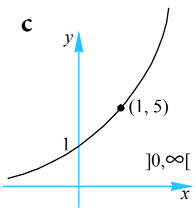
1 a



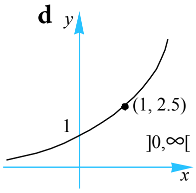
b



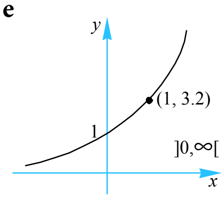
c



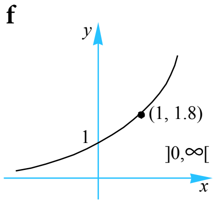
d



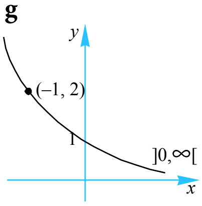
e



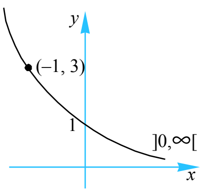
f



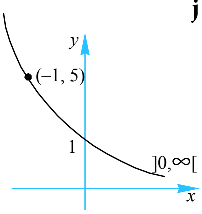
g



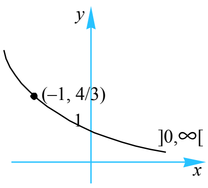
h



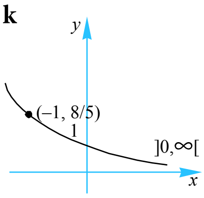
i



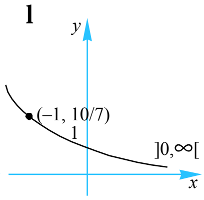
j



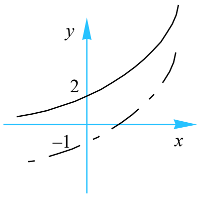
k



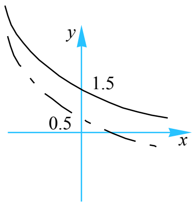
l



2 a

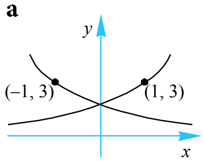


b

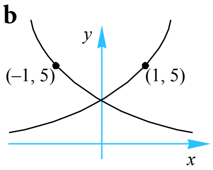


3 'b' has a dilation effect on $f(x) = a^x$ (along the y axis).

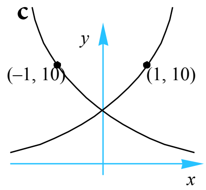
4 a



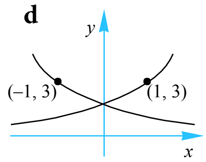
b



c

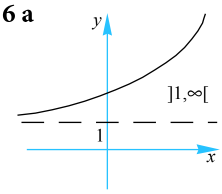


d

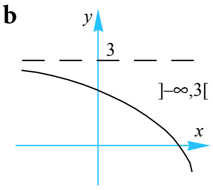


5 **a** [1,16] **b** [3,27] **c** [0.25,16] **d** [0.5,4] **e** [0.125,0.25] **f** [0.1,10]

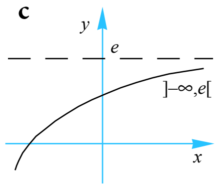
6 a



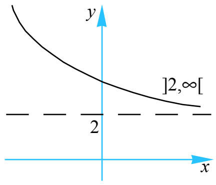
b



c

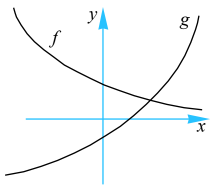


d



7 a –1. 5

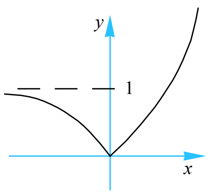
b



c i f = g: x = 1 **ii** f > g: x < 1

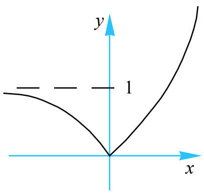
8 **a** $]2, 2 + e^{-1}[$ **b** $[-1, 1[$ **c** $[1 - e, 1 + e^{-1}]$

9 **a**

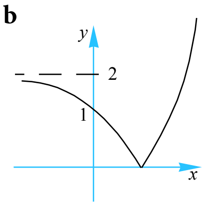


b 2

10 a



b



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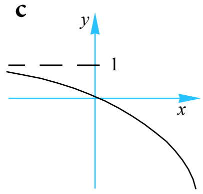
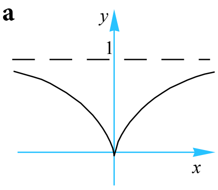


11 a



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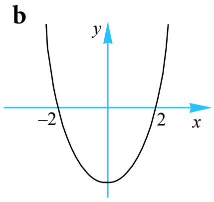

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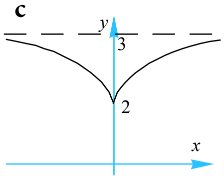
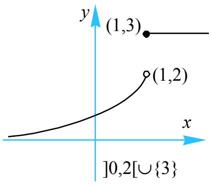


12 a



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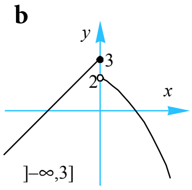

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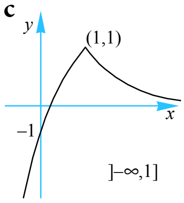

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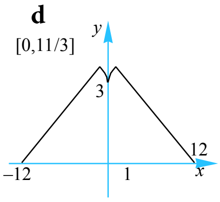
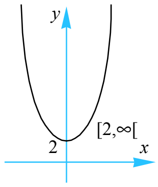


13 a



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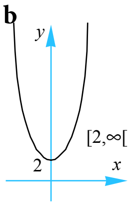

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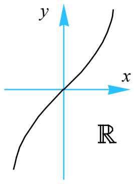

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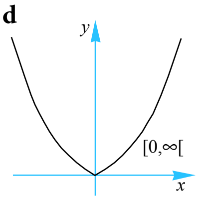
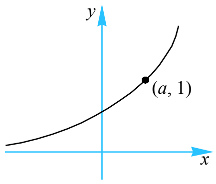


14 a



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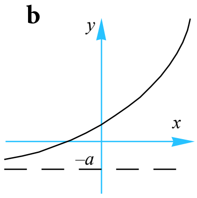

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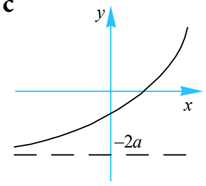

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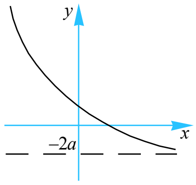
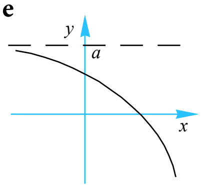


e



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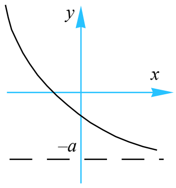
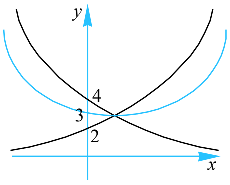


15 a



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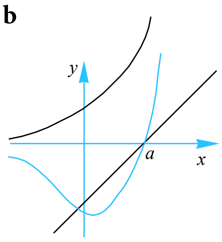
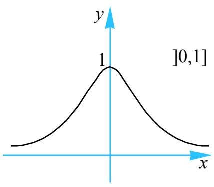


16 a



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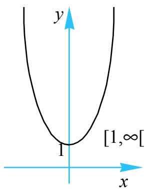

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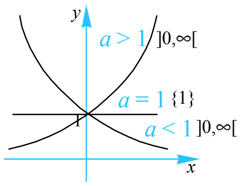

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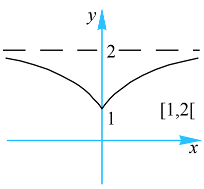

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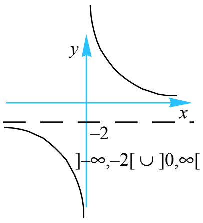

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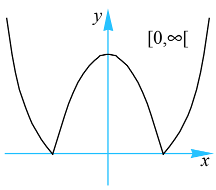
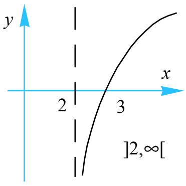


Exercise B.4.7

1 a



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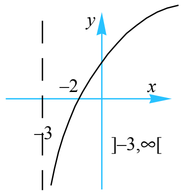

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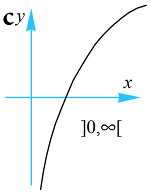

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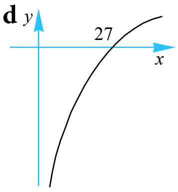

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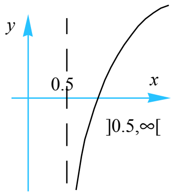

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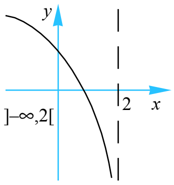

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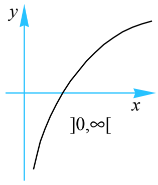

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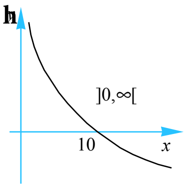

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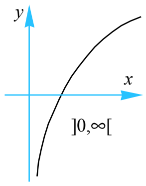

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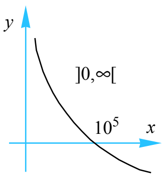

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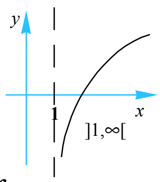

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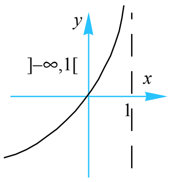

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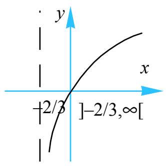

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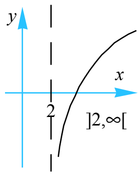

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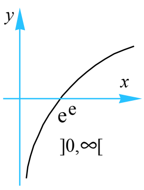

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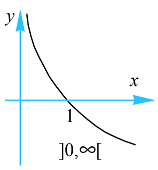

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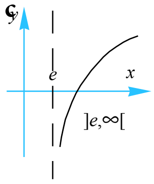

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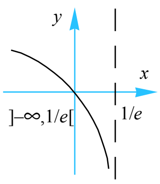

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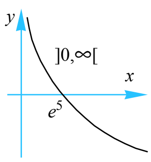

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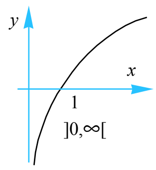

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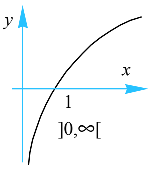

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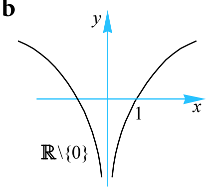

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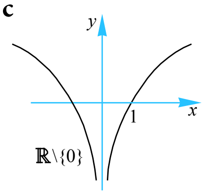

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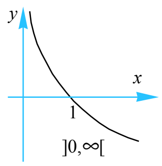

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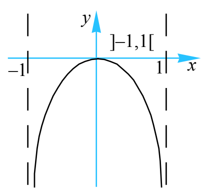

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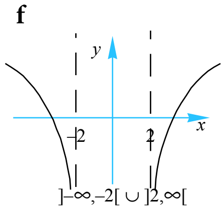

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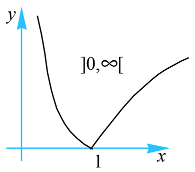

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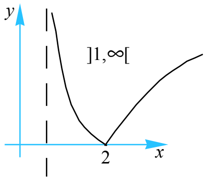

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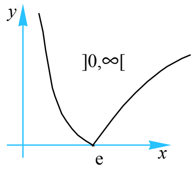

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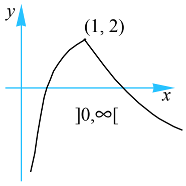

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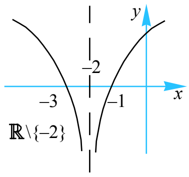

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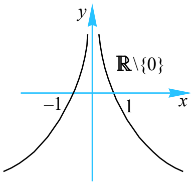
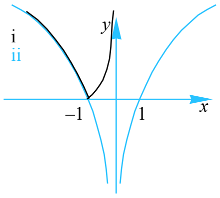


6 a



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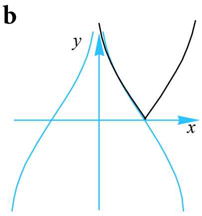

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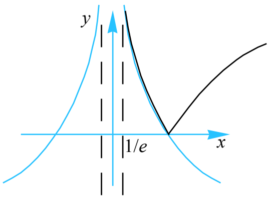

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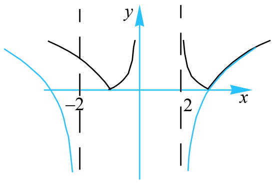

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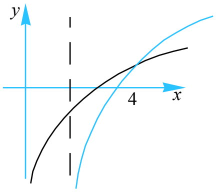
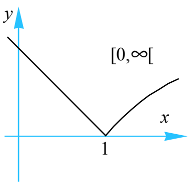


b $0 < x < \sim 4.3$

8 a



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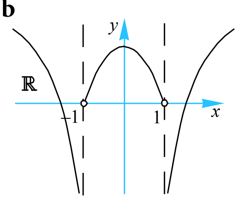

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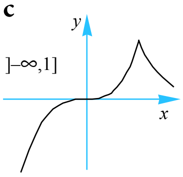

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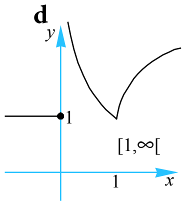
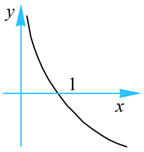


9 a



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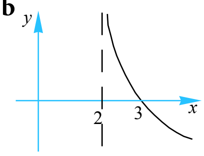

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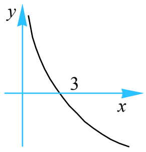

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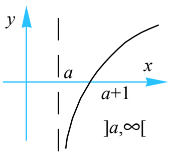

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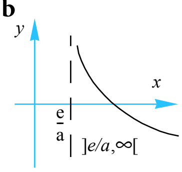

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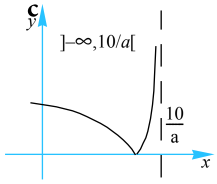

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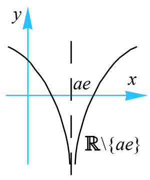

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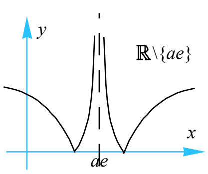

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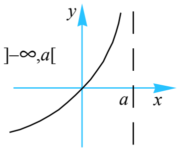
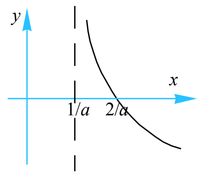
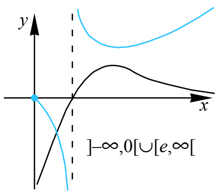


11 a



$$\left\{x : \frac{1}{a} < x < 1 + \frac{1}{a}\right\}$$

12

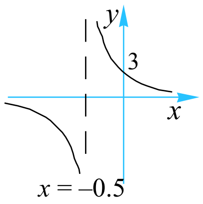


Exercise B.6.1

1 **a** $y = 2, x = -1$ **b** $y = 1, x = -\frac{1}{3}$ **c** $y = \frac{1}{2}, x = -\frac{1}{4}$

d $y = -1, x = -3$ **e** $y = 3, x = 0$ **f** $y = 5, x = 2$

3 a



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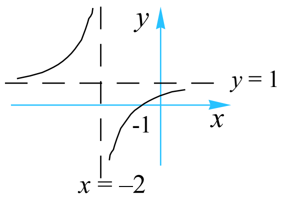

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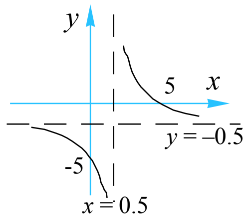

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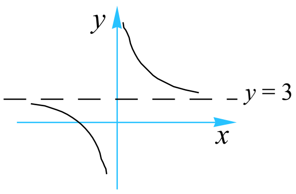

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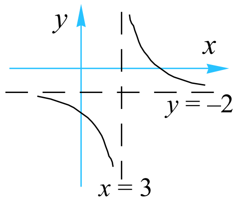

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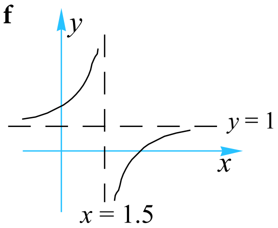
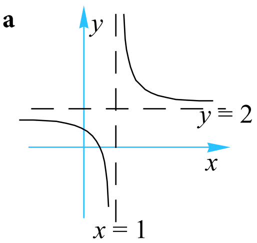


4 $a = 2, c = 4$

5 a



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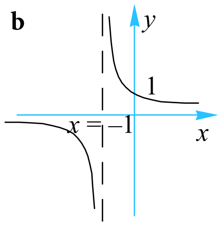
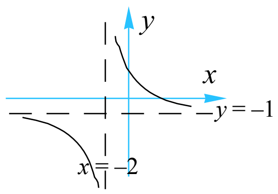
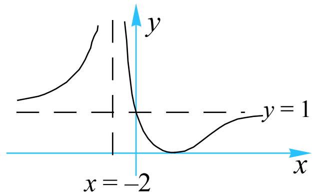


6 **a i** (0, 1), (2, 0) **ii** $y = -1, x = -2$ **iii**

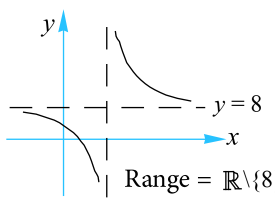


iv $d = \mathbb{R} \setminus \{-2\}$

b $f^{-1}: \mathbb{R}\setminus\{-1\} \mapsto \mathbb{R}$, where $f^{-1}(x) = \frac{2(1-x)}{(1+x)}$ **c**

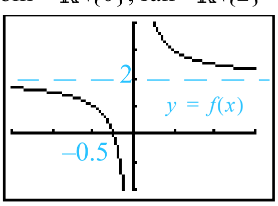


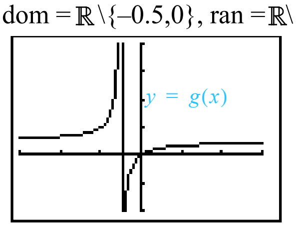
7 **a** $y = 8, x = 3$ **b**



Range = $\mathbb{R}\setminus\{8\}$

8 dom = $\mathbb{R}\setminus\{0\}$, ran = $\mathbb{R}\setminus\{2\}$ dom = $\mathbb{R}\setminus\{-0.5, 0\}$, ran = $\mathbb{R}\setminus\{0.5\}$





9 Asymptotes: **a** $y = 2x, x = 0$ **b** $y = \frac{1}{2}x, x = 0$ **c** $y = -x, x = 0$ **d** $y = x, x = 0$

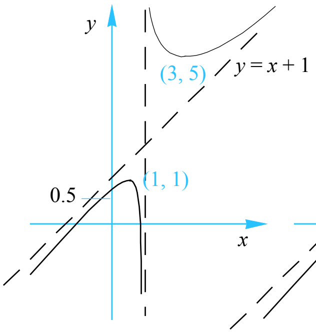
10 Asymptotes: **a** $y = x^2, x = 0$ **b** $y = x^2, x = 0$ **c** $y = x, x = 0$ **d** $y = x^3, x = 0$

11 Asymptotes: **a** $y = x+3, x = 0$ **b** $y = -x+2, x = 0$ **c** $y = 2x-2, x = 0$

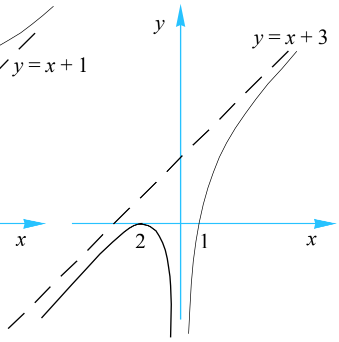
d $y = x+2, x = 0$

12 **a i** (0, 4), (2, 0) **ii** $y = 3-x, x = 1$

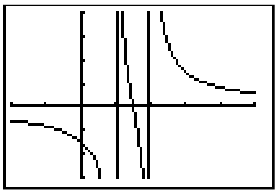
13 **a**



b



14 a



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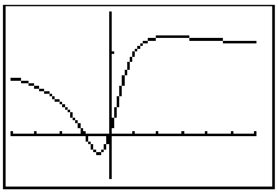
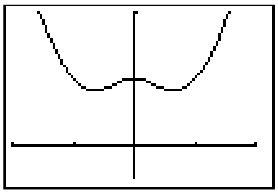
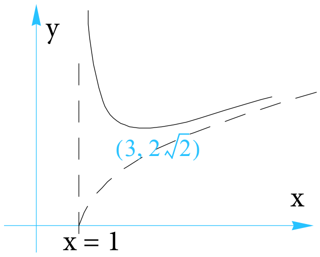


c

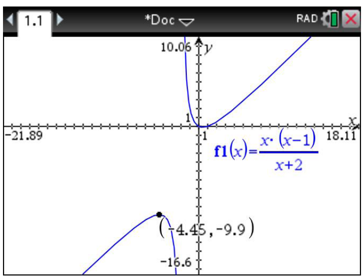


15

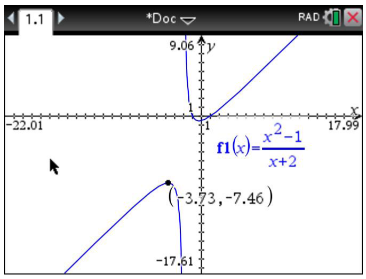


Exercise B.6.2

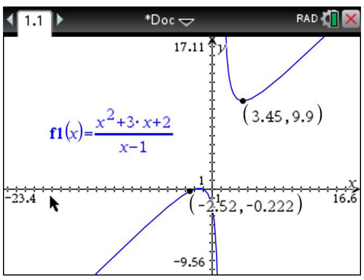
1. a



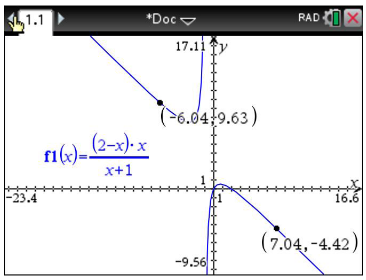
b



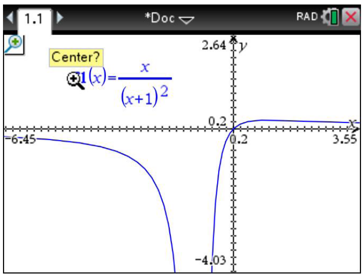
c



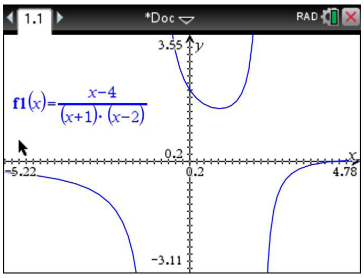
d



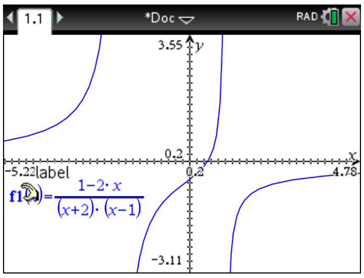
2. a



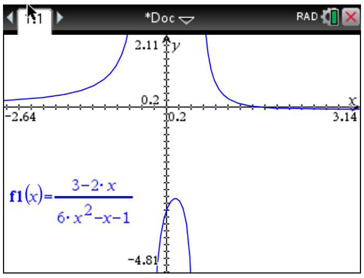
b



c



d



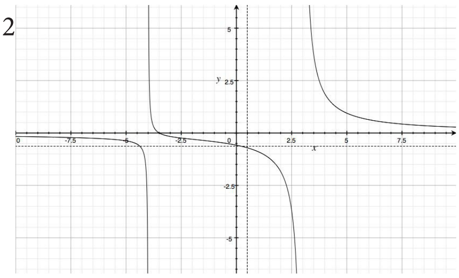
3. a $x=-4, y=x-5$ b $x=-\frac{1}{2},1, y=0$ c $x=-2,1, y=0$

d none

4. ⅓

5. (1,–1)

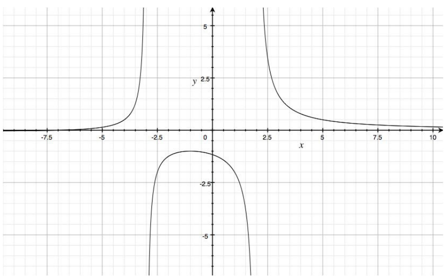
6. –12



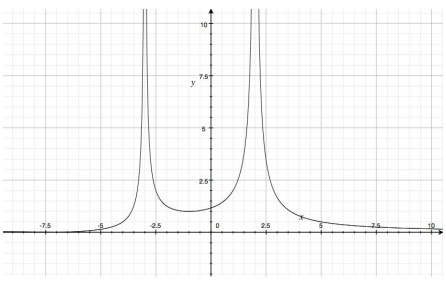
7. –4

8. 0

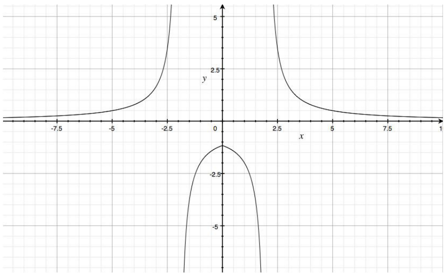
9. a



b



c



10. $(a,3a)$

11. ~3140

12. $p(\text{max}) = 8.88$ at $t = 2.82$

Exercise C.4.1

1

a $\frac{169\pi}{150}$ cm², $5.2+\frac{13\pi}{15}$ cm

b $\frac{529\pi}{32}$ cm², $23+\frac{23\pi}{8}$ cm

c 242π cm², $88 + 11\pi$ cm

d $\frac{1156\pi}{75}$ m², $13.6+\frac{68\pi}{15}$ m

e $\frac{96\pi}{625}$ cm², $1.28+\frac{12\pi}{25}$ cm

f $\frac{361\pi}{15}$ cm², $15.2+\frac{19\pi}{3}$ cm

g 5248.8π m², $648 + 32.4\pi$ cm

h $\frac{12943\pi}{300}$ cm², $17.2+\frac{301\pi}{30}$ cm

i $\frac{1922\pi}{75}$ cm², $12.4+\frac{124\pi}{15}$ cm

j $\frac{15884\pi}{3}$ cm², $152+\frac{418\pi}{3}$ cm

k 12π cm², $24 + 2\pi$ cm

l $\frac{98\pi}{3}$ cm², $28+\frac{14\pi}{3}$ cm

m $\frac{196\pi}{75}$ cm², $5.6+\frac{28\pi}{15}$ cm

n $\frac{11532\pi}{25}$ cm², $49.6+\frac{186\pi}{5}$ cm

o $\frac{3\pi}{50}$ cm², $2.4+\frac{\pi}{10}$ cm

2 0.63^c, 36°

3 0.0942 m³

4 1.64^c

5 79 cm

6 5.25 cm²

7 $\frac{\sqrt{50}\pi}{5}^c$

8 **a** 31.83 m **b** 406.28 m **c** 11°

9 1.11^c

10 0.75^c

11 **a** 1.85^c **b i** 37.09 cm **ii** 88.57 cm **c** 370.92 cm²

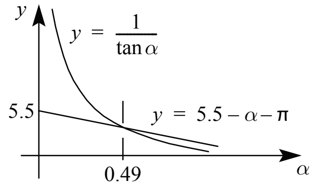
12 26.57 cm²

13 193.5 cm

14 **a** 105.22 cm **b** 118.83 cm

15 **a** 9 cm **b** 12 cm **c** 36° 52'

16 **b**



c 0.49

17 1439.16 cm^2

Exercise C.5.1

1	a	120°	b	108°	c	216°	d	50°
2	a	π^c	b	$\frac{3\pi^c}{2}$	c	$\frac{7\pi^c}{9}$	d	$\frac{16\pi^c}{9}$
3	a	$\frac{\sqrt{3}}{2}$	b	$-\frac{1}{2}$	c	$-\sqrt{3}$	d	-2
	e	$-\frac{1}{2}$	f	$-\frac{\sqrt{3}}{2}$	g	$\frac{1}{\sqrt{3}}$	h	$\sqrt{3}$
	i	$-\frac{1}{\sqrt{2}}$	j	$-\frac{1}{\sqrt{2}}$	k	1	l	$-\sqrt{2}$
	m	$-\frac{1}{\sqrt{2}}$	n	$\frac{1}{\sqrt{2}}$	o	-1	p	$\sqrt{2}$
	q	0	r	1	s	0	t	undefined
4	a	0	b	-1	c	0	d	-1
	e	$\frac{1}{\sqrt{2}}$	f	$-\frac{1}{\sqrt{2}}$	g	-1	h	$\sqrt{2}$
	i	$-\frac{1}{2}$	j	$-\frac{\sqrt{3}}{2}$	k	$\frac{1}{\sqrt{3}}$	l	$\sqrt{3}$
	m	$-\frac{\sqrt{3}}{2}$	n	$\frac{1}{2}$				
5	a	$\frac{1}{2}$	b	$\frac{\sqrt{3}}{2}$	c	11	d	$\frac{1}{2}$
	e	$-\frac{1}{\sqrt{3}}$	f	$-\frac{1}{2}$	g	$-\sqrt{2}$		
6	a	$-\frac{1}{2}$	b	$-\frac{1}{\sqrt{2}}$	c	$\sqrt{3}$	d	-2
	e	1	f	$\frac{1}{2}$	g	$-\frac{1}{\sqrt{3}}$	h	$-\frac{\sqrt{3}}{2}$
	i	$-\frac{2}{\sqrt{3}}$	j	$\frac{1}{\sqrt{3}}$	k	$\frac{2}{\sqrt{3}}$		

7	**a**	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	**b**	$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	**c**	$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	**d**	$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
8	**a**	0	**b**	$\frac{\sqrt{3}}{2}$	**c**	$\frac{1}{\sqrt{3}}$	**d**	$\frac{1+\sqrt{3}}{2\sqrt{2}}$
10	**a**	$-\frac{2}{3}$	**b**	$-\frac{2}{3}$	**c**	$-\frac{2}{3}$		
11	**a**	$-\frac{2}{5}$	**b**	$\frac{5}{2}$	**c**	$\frac{2}{5}$		
12	**a**	k	**b**	$-\frac{1}{k}$	**c**	$-k$		
13	**a**	$\frac{\sqrt{5}}{3}$	**b**	$\frac{3}{\sqrt{5}}$	**c**	$-\frac{\sqrt{5}}{3}$		
14	**a**	$-\frac{3}{5}$	**b**	$\frac{3}{4}$	**c**	$\frac{4}{5}$		
15	**a**	$\frac{4}{5}$	**b**	$\frac{3}{4}$	**c**	$-\frac{5}{3}$		
16	**a**	$-k$	**b**	$-\sqrt{1-k^2}$	**c**	$-\frac{k}{\sqrt{1-k^2}}$		
17	**a**	$-\sqrt{1-k^2}$	**b**	$\frac{k}{\sqrt{1-k^2}}$	**c**	$-\frac{1}{\sqrt{1-k^2}}$		
18	**a**	$\sin\theta$	**b**	$\cot\theta$	**c**	1	**d**	1
	e	$\cot\theta$	**f**	$\tan\theta$				
19	**a**	$\frac{\pi}{3}, \frac{2\pi}{3}$	**b**	$\frac{\pi}{3}, \frac{5\pi}{3}$	**c**	$\frac{\pi}{3}, \frac{4\pi}{3}$	**d**	$\frac{5\pi}{6}, \frac{7\pi}{6}$
	e	$\frac{5\pi}{6}, \frac{11\pi}{6}$	**f**	$\frac{7\pi}{6}, \frac{11\pi}{6}$				

Exercise C.5.2

3 **a** $x^2 + y^2 = k^2, -k \le x \le k$ **b** $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, -b \le x \le b$

c $(x-1)^2 + (2-y)^2 = 1, 0 \le x \le 2$ **d** $\frac{(1-x)^2}{b^2} + \frac{(y-2)^2}{a^2} = 1$

e $5x^2 + 5y^2 + 6xy = 16$

4 **a i** $-\frac{4}{5}$ **ii** $-\frac{5}{3}$ **b i** $\frac{4}{\sqrt{7}}$ **ii** $-\frac{\sqrt{7}}{3}$

5 **a** $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ **b** $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **c** $0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, 2\pi$ **d** $\frac{\pi}{2}, \frac{3\pi}{2}$

9 **a** $\frac{2a}{a^2+1}$ **b** $\frac{a^2-1}{a^2+1}$

10 **a i** 1 **ii** 1 **b** 1

11 **a** $\frac{1-\sqrt{x^2-1}}{x}$ **b** $\frac{1+\sqrt{x^2-1}}{x}$ **c** $\frac{2}{x^2} - 1$

12 **a i** 6 **ii** $\frac{5}{2}$ **iii** $\frac{9}{8}$

b i 5 **ii** 1 **iii** -2

13 **a** ± 2 **b** $\frac{\pi}{6} + 2k\pi, k \in \mathbb{Z}$ or $\frac{7\pi}{6} + 2k\pi, k \in \mathbb{Z}$

14 **a i** 25 **ii** $\frac{1}{5^4}$ **b i** 27 **ii** $\frac{1}{3}$

15 **a** $1 + 2k$ **b** $(1-k)\sqrt{1+2k}$

16 **a** $\frac{1-a}{2\sqrt{a}}$ **b i** $2 + \sqrt{2a - a^2}$ **ii** $\frac{-\sqrt{2a-a^2}}{1-a}$

17 **a** $\frac{2}{3}$ **b** $0, \pm\frac{2\sqrt{2}}{3}$

18 $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$

Exercise C.5.3

1 **a** $\sin\alpha\cos\phi + \cos\alpha\sin\phi$ **b** $\cos3\alpha\cos2\beta - \sin3\alpha\sin2\beta$ **c** $\sin2x\cos y - \cos2x\sin y$

d $\cos\phi\cos2\alpha + \sin\phi\sin2\alpha$ **e** $\frac{\tan2\theta - \tan\alpha}{1 + \tan2\theta\tan\alpha}$ **f** $\frac{\tan\phi - \tan3\omega}{1 + \tan\phi\tan3\omega}$

2 **a** $\sin(2\alpha - 3\beta)$ **b** $\cos(2\alpha + 5\beta)$ **c** $\sin(x + 2y)$

d $\cos(x - 3y)$ **e** $\tan(2\alpha - \beta)$ **f** $\tan x$

g $\tan\left(\frac{\pi}{4} - \phi\right)$ **h** $\sin\left(\frac{\pi}{4} + \alpha + \beta\right)$ **i** $\sin2x$

3 **a** $-\frac{56}{65}$ **b** $\frac{33}{65}$ **c** $-\frac{16}{63}$

4 **a** $\frac{16}{65}$ **b** $\frac{63}{65}$ **c** $\frac{56}{33}$

5 **a** $-\frac{5\sqrt{11}}{18}$ **b** $-\frac{7}{18}$ **c** $\frac{5\sqrt{11}}{7}$ **d** $\frac{35\sqrt{11}}{162}$

6 **a** $-\frac{3}{5}$ **b** $-\frac{4}{5}$ **c** $\frac{3}{4}$ **d** $\frac{24}{7}$

7 **a** $\frac{1 + \sqrt{3}}{2\sqrt{2}}$ **b** $\frac{1 + \sqrt{3}}{2\sqrt{2}}$ **c** $-\frac{1 + \sqrt{3}}{2\sqrt{2}}$ **d** $\sqrt{3} - 2$

8 **a** $\frac{2ab}{a^2 + b^2}$ **b** $\frac{a^2 + b^2}{2ab}$ **c** $\frac{a^4 - 6a^2b^2 + b^4}{(a^2 + b^2)^2}$ **d** $\frac{2ab}{b^2 - a^2}$

12 $\sqrt{2} - 1$

14 **a** $0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$ **b** $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$ **c** $0, \pi, 2\pi, \alpha, \pi \pm \alpha, 2\pi - \alpha, \alpha = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$

15 **a** $R = \sqrt{a^2 + b^2}, \tan\alpha = \frac{b}{a}$ **b** 10

16 **a** $R = \sqrt{a^2 + b^2}, \tan\alpha = \frac{b}{a}$ **b** -11

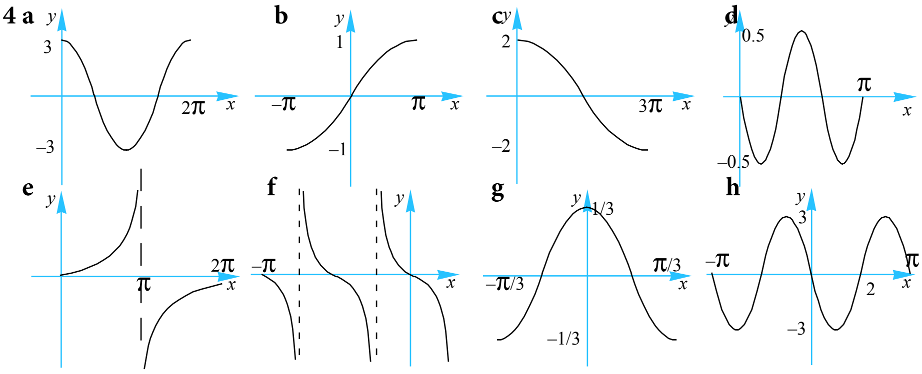
18 $2 - \sqrt{3}$

Exercise C.6.1

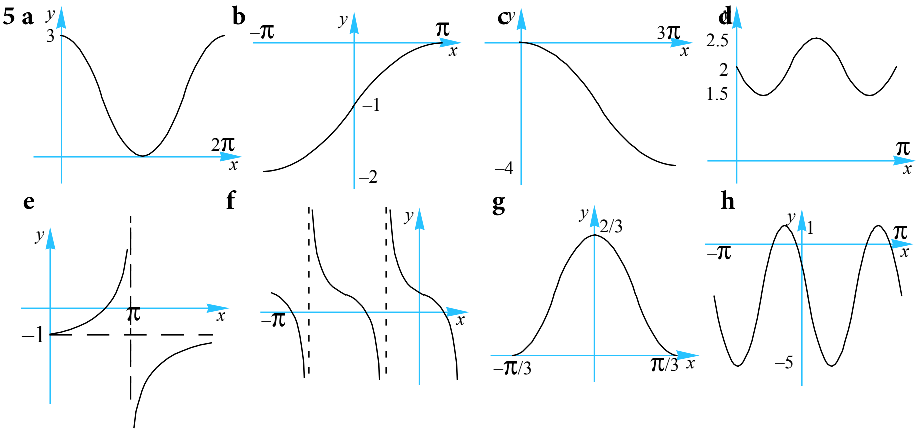
1	**a**	4π	**b**	$\frac{2\pi}{3}$	**c**	3π	**d**	4π	**e**	2	**f**	$\frac{\pi}{2}$
2	**a**	5	**b**	3	**c**	5	**d**	0.5				

3	**a**	2π, 2	**b**	6π, 3	**c**	π	**d**	π	**e**	π, 4
	f	π, 3	**g**	6π	**h**	$\frac{2\pi}{3}, \frac{1}{4}$	**i**	3π	**j**	$\frac{8\pi}{3}, \frac{2}{3}$

4



5



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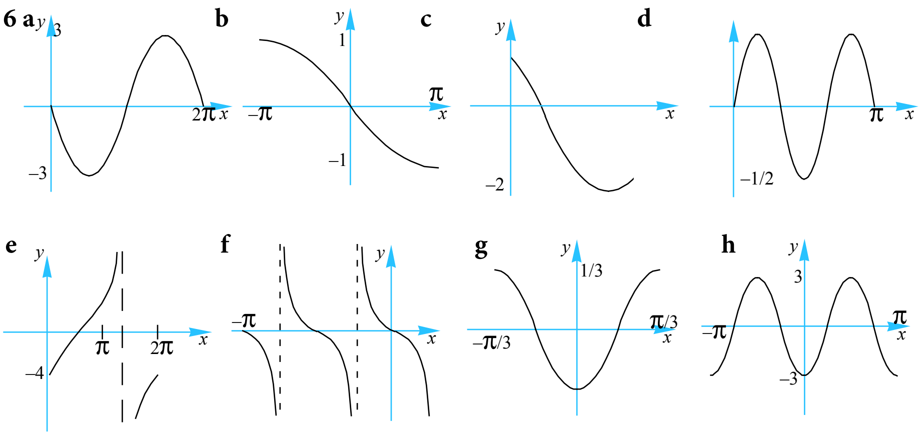

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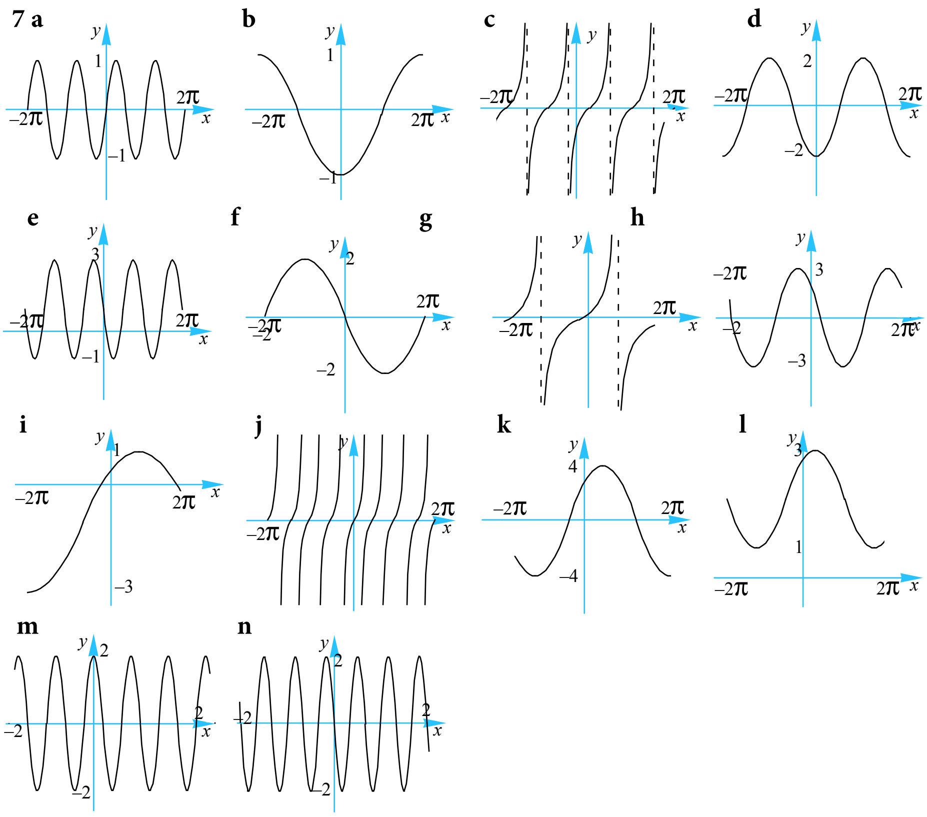
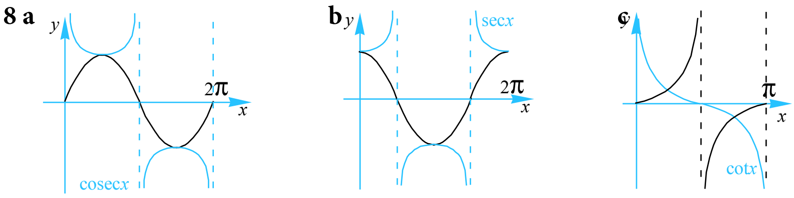

a
b
c
d
e
f
g
h
i
j
k
l
m
n
y
x
−2π
2π
1
−1
2
−2
3
−3
4
−4

8


a
b
c
y
x
2π
π
cosec$x$
sec$x$
cot$x$

Exercise C.7.1

1	**a**	$\frac{\pi}{4}, \frac{3\pi}{4}$	**b**	$\frac{7\pi}{6}, \frac{11\pi}{6}$	**c**	$\frac{\pi}{3}, \frac{2\pi}{3}$	**d**	$\frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}$
	e	$\frac{\pi}{3}, \frac{5\pi}{3}$	**f**	$\frac{5}{4}, \frac{7}{4}, \frac{13}{4}, \frac{15}{4}, \frac{21}{4}, \frac{23}{4}$				
2	**a**	$\frac{\pi}{4}, \frac{7\pi}{4}$	**b**	$\frac{2\pi}{3}, \frac{4\pi}{3}$	**c**	$\frac{\pi}{6}, \frac{11\pi}{6}$	**d**	π
	e	$\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$	**f**	$\frac{3}{2}, \frac{5}{2}, \frac{11}{2}$				
3	**a**	$\frac{\pi}{6}, \frac{7\pi}{6}$	**b**	$\frac{3\pi}{4}, \frac{7\pi}{4}$	**c**	$\frac{\pi}{3}, \frac{4\pi}{3}$	**d**	$4\tan^{-1}2$
	e	$\frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}$	**f**	3				
4	**a**	90°, 330°	**b**	180°,240°	**c**	90°,270°	**d**	65°,335°
	e	$\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$	**f**	$0, \pi, 2\pi$	**g**	$\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$	**h**	$\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$
5	**a**	60°,300°	**b**	$\frac{4\pi}{3}, \frac{5\pi}{3}$	**c**	$\frac{\pi}{6}, \frac{7\pi}{6}$	**d**	23°35',156°25'
	e	$\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$	**f**	$\frac{2\pi}{3}, \frac{5\pi}{3}$	**g**	$\frac{5\pi}{6}, \frac{9\pi}{6}$	**h**	$3.3559^c, 5.2105^c$
	i	$\frac{\pi}{3}, \frac{4\pi}{3}$	**j**	$\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$	**k**	$\frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$	**l**	68°12',248°12'
	m	$\frac{\pi}{3}, \frac{5\pi}{3}$	**n**	$\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$	**o**	Ø		
6	**a**	$-\frac{3\pi}{4}, \frac{\pi}{4}$	**b**	$\pm\frac{\pi}{3}$	**c**	$-\frac{7\pi}{8}, -\frac{3\pi}{8}, \frac{\pi}{8}, \frac{5\pi}{8}$	**d**	$-\frac{\pi}{2}$
	e	$\pm\frac{\pi}{2}$	**f**	$\frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}$	**g**	$\frac{\pi}{2}, \frac{3\pi}{2}$	**h**	$\frac{\pi}{2}, \frac{3\pi}{2}$

7 **a** $\frac{\pi}{6},\frac{5\pi}{6},\frac{3\pi}{2},\frac{13\pi}{6},\frac{17\pi}{6},\frac{7\pi}{2}$ **b** $-2\pi,0,2\pi$

c $-\frac{\pi}{2},\frac{3\pi}{2},\frac{7\pi}{2}$ **d** $\frac{\pi}{6},\frac{5\pi}{6},\frac{3\pi}{2},\frac{13\pi}{6},\frac{17\pi}{6},\frac{25\pi}{6},\frac{29\pi}{6}$

e $2n\pi \pm \sin^{-1}\left(\frac{1}{3}\right) \pm \frac{\pi}{2}, \frac{2(3n\pm1)\pi}{3}, n=-1,3$

8 **a** $\frac{3\pi}{4}, \frac{7\pi}{4}, \tan^{-1}\left(\frac{2}{3}\right), \pi + \tan^{-1}\left(\frac{2}{3}\right)$ **b** $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{4}$

9 **a** $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$ **b** $\frac{2\pi}{3}, \frac{4\pi}{3}$ **c** $0,1,2,3,4,5,6$

10 **a** $\frac{\pi}{3}, \frac{5\pi}{3}, \pi \pm \cos^{-1}\left(\frac{1}{4}\right)$ **b** $\frac{3\pi}{4}, \frac{7\pi}{4}, \tan^{-1}(3), \pi + \tan^{-1}(3)$ **c** $\frac{\pi}{6}, \frac{7\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{2}$

d $\tan^{-1}\left(\frac{3}{2}\right), \pi - \tan^{-1}(2), \pi + \tan^{-1}\left(\frac{3}{2}\right), 2\pi - \tan^{-1}(2)$

11 **a** $2\sin\left(x + \frac{\pi}{6}\right)$ **b** $0, \frac{2\pi}{3}, 2\pi$

12 **a** $2\sin\left(x - \frac{\pi}{3}\right)$ **b** $\frac{\pi}{6}, \frac{3\pi}{2}$

13 $\frac{\pi}{3}, \frac{2\pi}{3}$

14 **a** $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right) \cup \left(\frac{13\pi}{6}, \frac{17\pi}{6}\right)$ **b** $\left(\pi + \sin^{-1}\left(\frac{1}{\sqrt{3}}\right), 2\pi - \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)\right) \cup \left(3\pi + \sin^{-1}\left(\frac{1}{\sqrt{3}}\right), 4\pi - \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)\right)$

15 **a ii** $\left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$ **b ii** $\left[0, \frac{\pi}{6}\right) \cup \left(\frac{\pi}{2}, \frac{5\pi}{6}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right]$

17 **a i** $\{x | x = k\pi + \alpha(-1)^k, k \in \mathbb{Z}\}$ **ii** $\{x | 2k\pi + \alpha \le x \le (2k+1)\pi - \alpha, k \in \mathbb{Z}\}$

b $\left\{x | x = (2k+1)\frac{\pi}{5}\right\} \cup \{x | x = 2k\pi\}, k \in \mathbb{Z}$ **c** $\left\{x | x = \frac{2k\pi}{5} + \frac{\pi}{10}\right\} \cup \left\{x | x = 2k\pi - \frac{\pi}{2}\right\}, k \in \mathbb{Z}$

19 **a** $0, \frac{\pi}{3}, \frac{5\pi}{3}, 2\pi$ **b** $\sqrt{2}, \frac{\sqrt{2}}{2}$ **c** $2\cos\frac{\pi}{9}, 2\cos\frac{5\pi}{9}, 2\cos\frac{7\pi}{9}$

20 $\left\{\pm\frac{\pi}{4}, \pm\frac{2\pi}{3}, \pm\frac{3\pi}{4}\right\}$

22 **a** 90°,199°28',340°32' **b** (199°28',340°32')

25 $\left\{(x, y)|x = 2k\pi + \frac{\pi}{2}, y = 2k\pi\right\} \cup \left\{(x, y)|x = 2k\pi - \frac{\pi}{2}, y = 2k\pi + \pi\right\}, k \in \mathbb{Z}$

Exercise C.7.2

1 **a** 5, 24, 11, 19 **b** $T = 5\sin\left(\frac{\pi t}{12} - 3\right) + 19$ **c** 23.6°

2 **a** 3, 4.2, 2, 7 **b** $L = 3\sin\left(\frac{\pi t}{2.1} - 3\right) + 7$

3 **a** 5, 11, 0, 7 **b** $V = 5\sin\left(\frac{2\pi t}{11}\right) + 7$

4 **a** 1, 11, 1, 12 **b** $P = \sin\frac{2\pi}{11}(t - 1) + 12$

5 **a** 2.6, 7, 2, 6 **b** $S = 2.6\sin\frac{2\pi}{7}(t - 2) + 6$

6 **a** 0.6, 3.5, 0, 11 **b** $P = 0.6\sin\left(\frac{4\pi t}{7}\right) + 11$

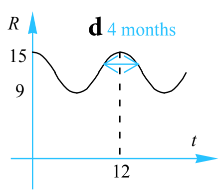
7 **a** 0.8, 4.6, 2.7, 11 **b** $D = 0.8\sin\frac{\pi}{2.3}(t - 2.7) + 11$

8 **a** 3000 **b** 1000, 5000 **c** $\frac{4}{9}$

9 **a** 6.5 m, 7.5 m **b** 1.58 sec, 3.42 sec

10 **a** 750, 1850 **b** 3.44 **c** mid-April to end of August

11 **a** 15000 **b** 12 months

c 

d 4 months

12 **a** 2s **b** 26cm **c** 40s

d [18,34] **d** 8cm **e** 2s

f $D(t)=8\sin\left(\pi\left(x+\frac{1}{2}\right)\right)+26$ (for example) **g** 34cm

13 **a** $D(t)=20\sin\left(\frac{5\pi}{6}(x+0.2)\right)+52$ (for example) **b** 72cm

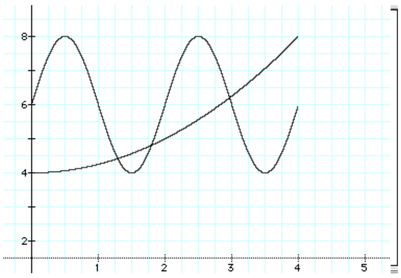
c 62cm **d** 0.86s

14 **a** π, −2, 2 **b** $\frac{1}{3}$m **c** $\frac{4}{3}$m

15 a

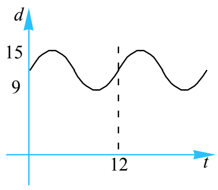
t	0	0.5	1	1.5	2	2.5	3	3.5	4
$F(t)$	6	8	6	4	6	8	6	4	6
$G(t)$	4	4.0625	4.25	4.5625	5	5.5625	6.25	7.0625	8

b



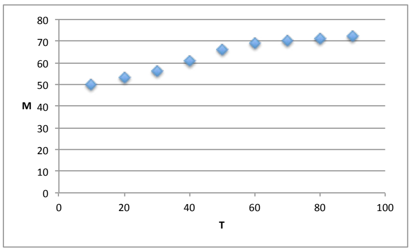
c 3 **d** 38.4%

16 **a**

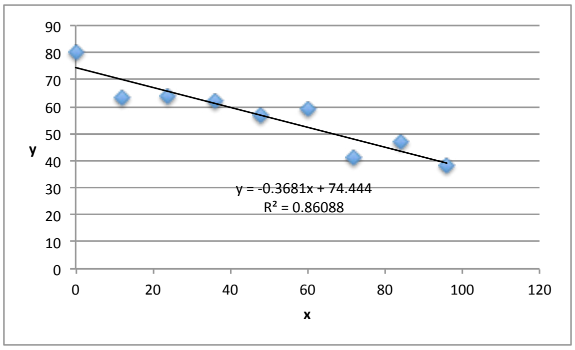


b i 7,11,19,23 **ii** $[0, 7] \cup [11, 19] \cup [23, 24]$ **c** 14.9 m

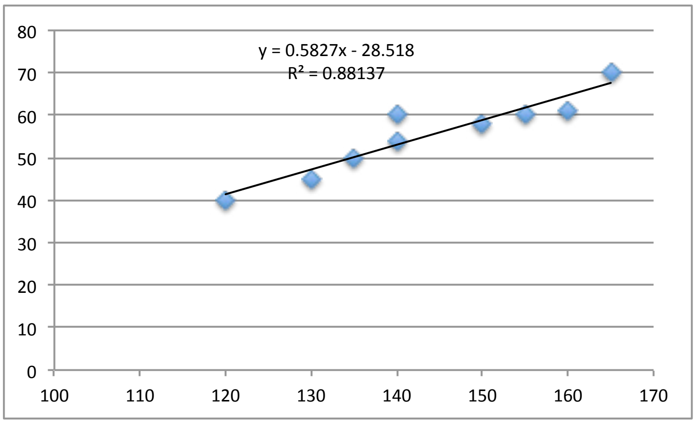
Exercise D.4.1

1. a 

b $r = 0.97$ c $M = 0.2967T + 48.28$

2. a, b, c 

d $y(40)=59.722$ (interpolation); $y(120) = 30.278$ (extrapolation)

3. 

4. B

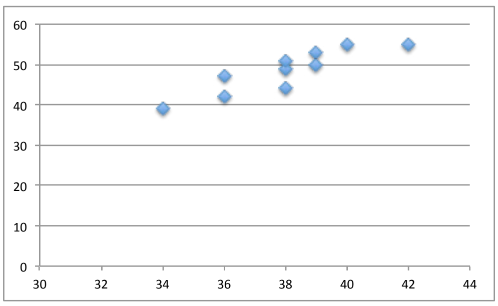
5. a 0.78 b i $P = 1.07M - 12.91$ ii 73%

c i $M = 0.77E + 27.14$ ii 100

iii Extrapolated. Continued linear trend highly likely. Therefore confident

d Find regression equation of E on M, then use $M = 90$ into this new equation.

6. a positive b linear c very strong

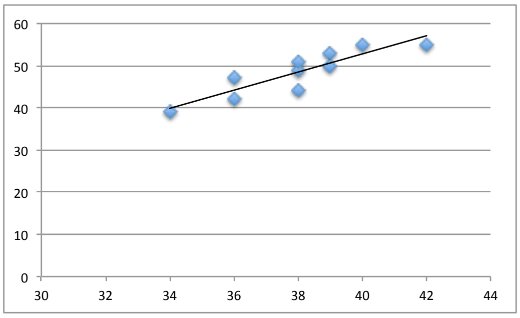
7 a i 

ii = 0.8908

b r^2 = 1.7935, that is, 79.35%

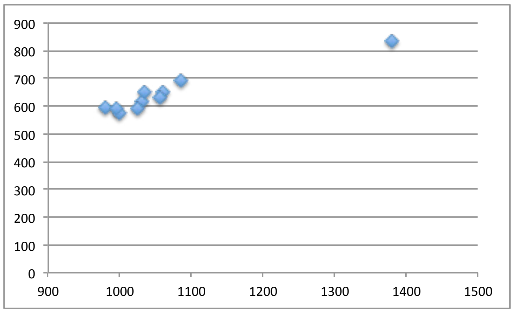
c $y = 2.15x - 33.28$

d



e $x = 37$, $y = 46.35$; Expenditure is $4635

8. a



ii $r = 0.9629$ b $y = 0.635x - 33.815$

d When $x = 1040$, $y = 626.59$. The carcass weighs 626.59 lbs.

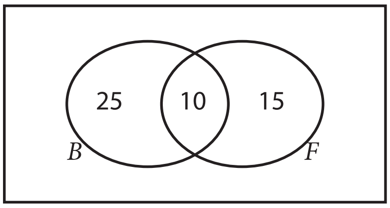
Exercise D.5.1

1. a 0.7 b 0.5 c 0.4

2. a 16 b $^{1}/_{16}$ c $^{3}/_{8}$ d $^{5}/_{16}$

3. a $^{1}/_{81}$ b $^{1}/_{72}$

4. a



b $^{1}/_{8}$ c $^{5}/_{16}$ d $^{3}/_{8}$

5. a $^{3}/_{4}$ b $^{3}/_{5}$ c 0.2 , 0.3 ; the events are not independent.

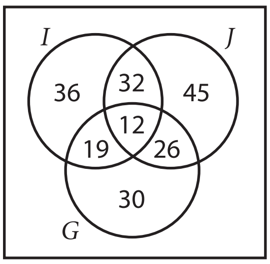
6. a $^{4}/_{51}$ b No c $^{1}/_{13}$

7. a i Yes ii $^{9}/_{64}$ iii ¼

   b i No ii $^{3}/_{28}$ iii $^{3}/_{14}$

8. a 0.76 b 0.24 c 0.6

9. a



b $^{3}/_{50}$ c $^{77}/_{200}$

d $^{44}/_{115}$

e No, selections made without replacement are always dependent.

10. a ½ b $^{1}/_{8}$ c $^{1}/_{16}$ d ½

    e $^{3}/_{8}$ f $^{7}/_{16}$ g $^{1}/_{4}$ h $^{1}/_{8}$

    i $^{1}/_{4}$

Toolbox notes

The Neglected Friend

What if the eastbound train arrives 'on the hour' and the westbound at 'five past the hour'?

The Gamow-Stern Elevator Paradox

Look at the number of floors below Harriet's floor. The probability that the elevator is below and heading up is larger that the probability that it is above and heading down. The reverse applies for the low floors. The situation improves if more elevators are added.

Crown and Anchor

The answer to this is to list all 216 outcomes and the payouts for each. You should find that for each $1 wagered 7.8 cents goes to the casino.

The Wallet Paradox

As far as we know, nobody has come up with a good explanation of this one!

Exercise D.6.1

1.	a	4.55	b	5.52	c	13.1	d	-1.88
	e	-5.1						
2.	a	2	b	33	c	1.6	d	-1
	e	-4.6						

3. 69.7 kg

4. 46 min 22 sec

5. Yes.

6. 1.0146 mm

7. a About 0.27% b About 4.6 times as many. c About one sixth.

Exercise D.6.2

1.	a	4	b	37	c	7.9	d	-1.2
	e	148						
2.	a	0.25	b	0.23	c	12	d	13
	e	0.95						
3.	a	$\mu = 5, \sigma = 0.9$	b	$\mu = 4.4, \sigma = 0.25$	c	$\mu = 34, \sigma = 1.3$	d	$\mu = 172, \sigma = 5$
	e	$\mu = -3, \sigma = 0.5$						

4. 2.3 gm.

5. a $\mu = 354$ mL, $\sigma = 2.3$ mL b 7 or 8

6. a $\mu = 450$ sec , $\sigma = 35$ sec b ~25

7. Before signs: $\mu = 55$ kph , $\sigma = 6$ kph. After signs: $\mu = 53$ kph , $\sigma = 4$ kph.

   There is some reduction in speed but also, after the signage, the cars are going at more consistent speeds (safer?).

Exercise E.2.1

1

a $-\frac{3}{x^4}$ **b** $\frac{3}{2}\sqrt{x}$ **c** $\frac{5}{2}\sqrt{x^3}$ **d** $\frac{1}{3\sqrt[3]{x^2}}$

e $\frac{2}{\sqrt{x}}$ **f** $9\sqrt{x}$ **g** $\frac{1}{\sqrt{x}}+\frac{3}{x^2}$ **h** $\frac{3}{2}\sqrt{x}-\frac{1}{2\sqrt{x^3}}$

i $\frac{10}{3\sqrt[3]{x}}-9$ **j** $5-\frac{1}{2\sqrt{x}}-\frac{8}{5x^3}$ **k** $\frac{4}{\sqrt{x}}-\frac{15}{x^6}+\frac{1}{2}$ **l** $-\frac{1}{2\sqrt{x^3}}-\frac{1}{\sqrt{x}}+x^2$

2

a $\frac{3}{2}\sqrt{x}+\frac{1}{\sqrt{x}}$ **b** $4x^3+3x^2-1$ **c** $3x^2+1$ **d** $\frac{1}{x^2}$

e $\frac{1}{\sqrt{x^3}}$ **f** $\frac{1}{2}-\frac{1}{4\sqrt{x^3}}$ **g** -7 **h** $2x-\frac{8}{x^3}$

i $2x-\frac{2}{x^2}-\frac{4}{x^5}$ **j** $\frac{1}{2}\sqrt{\frac{3}{x}}+\frac{1}{6\sqrt{x^3}}$ **k** $2x-\frac{12}{5}\sqrt[5]{x}+\frac{2}{5\sqrt[5]{x^3}}$

l $-\frac{3}{2\sqrt{x}}\left(\frac{1}{x}+1\right)\left(\frac{1}{\sqrt{x}}-\sqrt{x}\right)^2$

Exercise E.2.2

1

a $48t^3-\frac{1}{2\sqrt{t}}$ **b** $2n-\frac{2}{n^2}-\frac{4}{n^5}$ **c** $\frac{3}{2}\sqrt{r}+\frac{5}{6\sqrt[6]{r}}-\frac{1}{\sqrt{r}}$

d $2\theta-\frac{9}{2}\sqrt{\theta}+3-\frac{1}{2\sqrt{\theta}}$ **e** $40-3L^2$ **f** $-\frac{100}{v^3}-1$

g $6l^2+5$ **h** $2\pi+8h$

2

a $\frac{8}{3t^3}$ **b** $2\pi r-\frac{20}{r^2}$ **c** $\frac{5}{2}s^{3/2}+\frac{3}{s^2}$

d $-\frac{6}{t^4}+\frac{2}{t^3}-\frac{1}{t^2}$ **e** $-\frac{4}{b^2}+\frac{1}{2b^{3/2}}$ **f** $3m^2-4m-4$

Exercise E.2.3

1

a $3x^2-5x^4+2x+2$ **b** $6x^5+10x^4+4x^3-3x^2-2x$

c $-\frac{4}{x^5}$ **d** $6x^5+8x^3+2x$

2	**a**	$-\frac{2}{(x-1)^2}$	**b**	$\frac{1}{(x+1)^2}$	**c**	$\frac{1-x^2-2x}{(x^2+1)^2}$
	d	$\frac{-(x^4+3x^2+2x)}{(x^3-1)^2}$	**e**	$\frac{2x^2+2x}{(2x+1)^2}$	**f**	$\frac{1}{(1-2x)^2}$
3	**a**	$(\sin x+\cos x)e^x$	**b**	$\ln x+1$	**c**	$e^x(2x^3+6x^2+4x+4)$
	d	$4x^3\cos x-x^4\sin x$	**e**	$-\sin^2 x+\cos^2 x$	**f**	$2x\tan x+(1+x^2)\sec^2 x$
	g	$\frac{4}{x^3}(x\cos x-2\sin x)$	**h**	$e^x(x\cos x+x\sin x+\sin x)$		
	i	$(\ln x+1+x\ln x)e^x$				
4	**a**	$\frac{\sin x-x\cos x}{\sin^2 x}$	**b**	$\frac{-[\sin x(x+1)+\cos x]}{(x+1)^2}$	**c**	$\frac{e^x}{(e^x+1)^2}$
	d	$\frac{2x\cos x-\sin x}{2x\sqrt{x}}$	**e**	$\frac{\ln x-1}{(\ln x)^2}$	**f**	$\frac{(x+1)-x\ln x}{x(x+1)^2}$
	g	$\frac{xe^x+1}{(x+1)^2}$	**h**	$\frac{-2}{(\sin x-\cos x)^2}$	**i**	$\frac{x^2-x+2x\ln x}{(x+\ln x)^2}$
5	**a**	$-5e^{-5x}+1$	**b**	$4\cos 4x+3\sin 6x$	**c**	$-\frac{1}{3}e^{-\frac{1}{3}x}-\frac{1}{x}+18x$
	d	$25\cos 5x+6e^{2x}$	**e**	$4\sec^2 4x+2e^{2x}$	**f**	$-4\sin(4x)+3e^{-3x}$
	g	$\frac{4}{4x+1}-1$	**h**	0	**i**	$\frac{1}{2}\cos\left(\frac{x}{2}\right)-2\sin 2x$
	j	$7\cos(7x-2)$	**k**	$\frac{1}{2\sqrt{x}}-\frac{1}{x}$	**l**	$\frac{1}{x}+6\sin 6x$
6	**a**	$2x\cos x^2+2\sin x\cos x$	**b**	$2\sec^2 2\theta-\frac{\cos\theta}{\sin^2\theta}$	**c**	$\frac{1}{2\sqrt{x}}\cos\sqrt{x}$
	d	$\frac{1}{x^2}\sin\left(\frac{1}{x}\right)$	**e**	$-3\sin\theta\cdot\cos^2\theta$	**f**	$e^x\cos(e^x)$
	g	$\frac{1}{x}\sec^2(\log_e x)$	**h**	$\frac{-\sin 2x}{\sqrt{\cos 2x}}$	**i**	$-\cos\theta\cdot\sin(\sin\theta)$
	j	$4\sin\theta\cdot\sec^2\theta$	**k**	$-5\cos 5x\cdot\csc^2(5x)$	**l**	$-6\csc^2(2x)$

7 **a** $2e^{2x+1}$ **b** $-6e^{4-3x}$ **c** $-12xe^{4-3x^2}$

d $\frac{1}{2}\sqrt{e^x}$ **e** $\frac{1}{2\sqrt{x}}e^{\sqrt{x}}$ **f** e^{2x+4}

g $2xe^{2x^2+4}$ **h** $-\frac{6}{e^{3x+1}}$ **i** $(6x-6)e^{3x^2-6x+1}$

j $\cos(\theta)e^{\sin\theta}$ **k** $2\sin(2\theta)e^{-\cos 2\theta}$ **l** $2x$

m $\frac{2e^{-x}}{(e^{-x}+1)^2}$ **n** $3(e^x+e^{-x})(e^x-e^{-x})^2$ **o** e^{x+2}

p $(-2x+9)e^{-x^2+9x-2}$

8 **a** $\frac{2x}{x^2+1}$ **b** $\frac{\cos\theta+1}{\sin\theta+\theta}$ **c** $\frac{e^x+e^{-x}}{e^x-e^{-x}}$ **d** $-\frac{1}{x+1}$

e $\frac{3}{x}(\ln x)^2$ **f** $\frac{1}{2x\sqrt{\ln x}}$ **g** $\frac{1}{2(x-1)}$ **h** $\frac{-3x^2}{1-x^3}$

i $-\frac{1}{2(x+2)}$ **j** $\frac{-2\sin x\cos x}{\cos^2 x+1}$ **k** $\frac{1}{x}+\cot x$ **l** $\frac{1}{x}+\tan x$

9 **a** $\ln(x^3+2)+\frac{3x^3}{x^3+2}$ **b** $\frac{\sin^2 x}{2\sqrt{x}}+2\sqrt{x}\sin x\cos x$ **c** $-\frac{1}{\sqrt{\theta}}\sin\sqrt{\theta}\cdot\cos\sqrt{\theta}$

d $(3x^2-4x^4)e^{-2x^2+3}$ **e** $-(\ln x+1)\sin(x\ln x)$ **f** $\frac{1}{x\ln x}$

g $\frac{(2x-4)\cdot\sin(x^2)-2x\cdot\cos(x^2)(x^2-4x)}{(\sin x^2)^2}$ **h** $\frac{10(\ln(10x+1)-1)}{[\ln(10x+1)]^2}$

i $(\cos 2x-2\sin 2x)e^{x-1}$ **j** $2x\ln(\sin 4x)+4x^2\cot 4x$ **k** $(\cos\sqrt{x}-\sin\sqrt{x})\frac{1}{2\sqrt{x}}e^{-\sqrt{x}}$

l $-(2\sin x+2x\cos x)\cdot\sin(2x\sin x)$ **m** $\frac{e^{5x+2}(9-20x)}{(1-4x)^2}$ **n** $\frac{\cos^2\theta+\sin^2\theta\ln(\sin\theta)}{\sin\theta\cos^2\theta}$

o $\frac{x+2}{2(x+1)\sqrt{x+1}}$ **p** $\frac{2x^2+2}{\sqrt{x^2+2}}$ **q** $\frac{10x^3+9x^2+4x+3}{3(x+1)^{2/3}}$

r $\frac{3x^2(3x^3+1)}{2\sqrt{x^3+1}}$ **s** $\frac{2}{x^2+1}-\frac{1}{x^2}\ln(x^2+1)$ **t** $\frac{2}{x(x+2)}$

u $\frac{2-x}{2x^2\sqrt{x-1}}$ **v** $\frac{-x^2+x-9}{\sqrt{x^2+9}} \cdot e^{-x}$ **w** $\frac{7x^3-12x^2-8}{2\sqrt{2-x}}$

x $nx^{n-1}\ln(x^n-1)+\frac{nx^{2n-1}}{x^n-1}$

10 $x = 1$

11 0

12 0

13 1

14 $-2e$

15 **a** $\cos^2 x - \sin^2 x$ **b** $\frac{\pi}{180}\cos x°$ **c** $-\frac{\pi}{180}\sin x°$

16 **b i** $2x\sin x\cos x + x^2\cos^2 x - x^2\sin^2 x$ **ii** $e^{-x^3}(2\cos 2x\ln\cos x - 3x^2\sin 2x\ln\cos x - \sin 2x\tan x)$

17 **a i** $-\frac{3}{x}(\ln x)^2$ **ii** $-\frac{3x^2}{1-x^3}$ **b i** $-2e^{-2x} \cdot \cos(e^{-2x})$ **ii** $-2x\cos x^2 \cdot e^{-\sin x^2}$

18 $-\frac{1}{5}k$

19 $x = a, b, \frac{mb+na}{m+n}$

20 $\{\theta : n\tan\theta^m \cdot \tan\theta^n = m\theta^{m-n}\}$

21 **a** $-4\csc(4x)$ **b** $2\sec(2x)\tan(2x)$ **c** $3\cot(3x)\csc(3x)$ **d** $-3\sin(3x)$

e $\csc^2\left(\frac{\pi}{4}-x\right)$ **f** $-2\sec(2x)\tan(2x)$

22 **a** $2x\sec(x^2)\tan(x^2)$ **b** $\sec^2 x$ **c** $\tan x$

d $-3\cot^2 x\csc^2 x$ **e** $x\cos x + \sin x$ **f** $-2\cot x\csc^2 x$

g $4x^3\csc(4x) - 4x^4\cot(4x)\csc(4x)$ **h** $2\cot x\sec^2(2x) - \csc^2 x\tan(2x)$

i $\frac{\sec x\tan x - \sin x}{2\sqrt{\cos x + \sec x}}$

23 **a** $e^{\sec x}\sec x\tan x$ **b** $e^x\sec(e^x)\tan(e^x)$ **c** $e^x\sec(x) + e^x\sec(x)\tan(x)$

d $\frac{-\csc^2(\log x)}{x}$ **e** $-5\csc(5x)\sec(5x)$ **f** $\frac{\cot(x)}{x} - \csc^2(x)\log x$

g $-\cos x\cot(\sin x)\csc(\sin x)$ **h** $-\cos(\csc x)\cot x\csc x$ **i** 0

Exercise E.2.4

1 **a** $20x^3$ **b** $48(1+2x)^2$ **c** $\frac{2}{x^3}$ **d** $\frac{2}{(1+x)^3}$

e 2 **f** $\frac{6}{(x-2)^3}$ **g** $\frac{42}{x^8}$ **h** $24(1-2x)$

i $-\frac{1}{x^2}$ **j** $\frac{2(x^2+1)}{(1-x^2)^2}$ **k** $-16\sin 4\theta$ **l** $2\cos x - x\sin x$

m $6x^2\cos x + 6x\sin x - x^3\sin x$ **n** $\frac{1}{x}$ **o** $\frac{10}{(2x+3)^3}$

p $6xe^{2x} + 12x^2e^{2x} + 4x^3e^{2x}$ **q** $\frac{8\sin 4x - 15\cos 4x}{e^x}$

r $2\cos x^2 - 4x^2\sin x^2$ **s** $\frac{-48(x^2+2x^5)}{(4x^3-1)^3}$ **t** $\frac{10}{(x-3)^3}$

2 **a** $6x^2\cos x + (6x - x^3)\sin x$ **b** $2\ln x + 3$

c $\frac{-4}{(x+3)^3}$ **d** $(\cos^2 x - \sin x)e^{\sin x}$

e $\frac{-1}{4(x+1)\sqrt{x+1}}$ **f** $\frac{(4x^2+4x+2)e^{2x}}{(x+1)^3}$

3 $\frac{6\ln x - 5}{x^4}, \frac{n^2\ln x + n\ln x - 2n - 1}{x^{n+2}}$

4 $f'(x) = -\frac{1}{(x+1)^2}, f''(x) = \frac{2}{(x+1)^3}, f^{(iii)}(x) = -\frac{6}{(x+1)^4}, f^{(iv)}(x) = \frac{24}{(x+1)^5}$

$f^{(v)}(x) = -\frac{120}{(x+1)^6}, \ldots\ldots, f^{(n)}(x) = (-1)^n\frac{n!}{(x+1)^{n+1}}$

5 $f(x) = \left(\frac{x+1}{x-1}\right)^n \Rightarrow f''(x) = \frac{4n(n+x)}{(x^2-1)^2}\left(\frac{x+1}{x-1}\right)^n$

7 **a** $a^n e^{ax}$ **b** $\frac{(-1)^n 2^n n!}{(2x+1)^{n+1}}$

c $n = 2k: \quad y^{(n)}(x) = (-1)^k a^{2k}\sin(ax+b), k = 1, 2, \ldots$

$n = 2k-1: \quad y^{(n)}(x) = (-1)^{k+1}a^{2k-1}\cos(ax+b), k = 1, 2, \ldots$

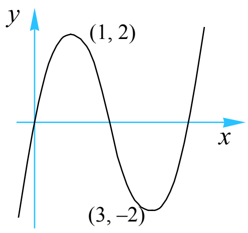
8	**a**	$2+\frac{1}{8\sqrt{2}}$	**b**	$\frac{3+\pi}{2}$
9	−1	[0,1.0768[∪]3.6436,2p]		

Exercise E.2.5

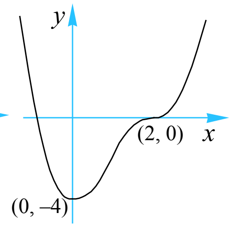
1.	a	203	b	2.25	c	2.6923
	d	15.91	e	23.9	f	1

Exercise E.4.1

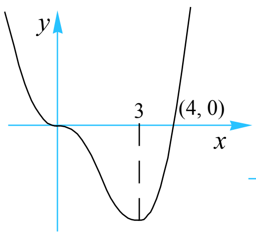
1 **a**



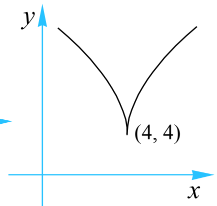
b



c



d



2 **a** max at (1, 4) **b** min at $\left(-\frac{9}{2}, -\frac{81}{4}\right)$ **c** min at (3, –45) max (–3, 63)

d max at (0, 8), min at (4, -24) **e** max at (1, 8), min at (–3, –24)

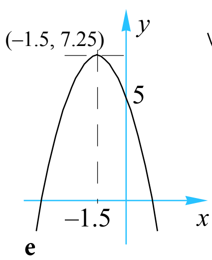
f min at $\left(\frac{1+\sqrt{13}}{3}, \frac{70-26\sqrt{13}}{27}\right)$, max at $\left(\frac{1-\sqrt{13}}{3}, \frac{70+26\sqrt{13}}{27}\right)$ **g** min at (1, –1)

h max at (0, 16), min at (2, 0), min at (–2, 0) **i** min at (1, 0) max at $\left(-\frac{1}{3}, \frac{32}{27}\right)$

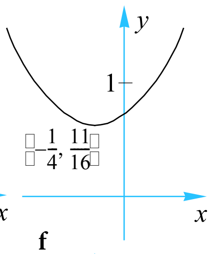
j min at $\left(\frac{4}{9}, -\frac{4}{27}\right)$ **k** min at (2, 4), max at (–2, –4)

l min at (1, 2), min at (–1, 2)

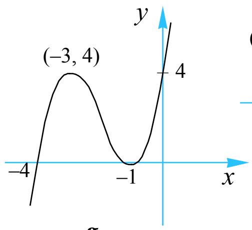
3 a



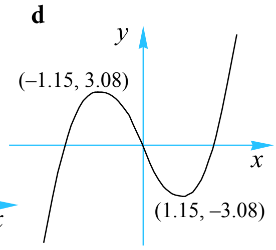
b



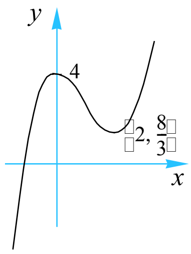
c



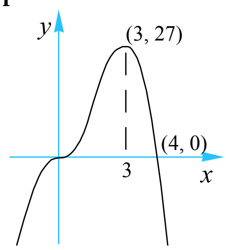
d



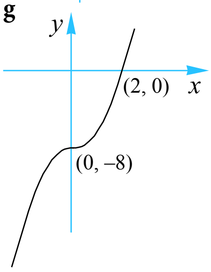
e



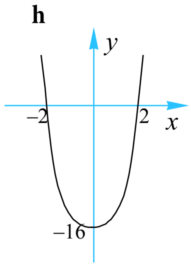
f



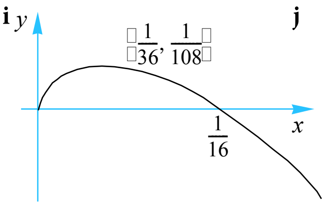
g



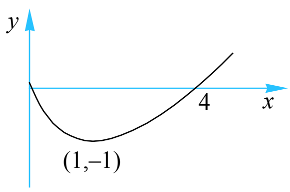
h



i

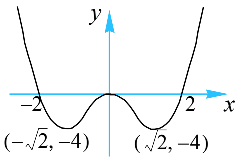


j



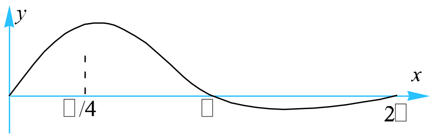
4 min at (1, –3), max at (–3, 29), non-stationary infl (–1, 13)

5



6 **a i** $(\cos x - \sin x)e^{-x}$ **ii** $-2\cos x.e^{-x}$ **b i** $\frac{\pi}{4}, \frac{5\pi}{4}$ **ii** $\frac{\pi}{2}, \frac{3\pi}{2}$

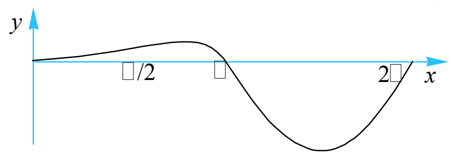
c Inf. $\left(\frac{\pi}{2}, e^{-\frac{\pi}{2}}\right), \left(\frac{3\pi}{2}, -e^{-\frac{3\pi}{2}}\right)$ **d**



7 **a i** $e^x(\sin x + \cos x)$ **ii** $2e^x\cos x$ **b i** $x = \frac{3\pi}{4}, \frac{7\pi}{4}$ **ii** $x = \frac{\pi}{2}, \frac{3\pi}{2}$

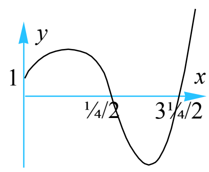
c St. pts. $\left(\frac{3\pi}{4}, \frac{1}{\sqrt{2}}e^{\frac{3\pi}{4}}\right), \left(\frac{7\pi}{4}, -\frac{1}{\sqrt{2}}e^{\frac{7\pi}{4}}\right)$ Infl. pts. $\left(\frac{\pi}{2}, e^{\frac{\pi}{2}}\right), \left(\frac{3\pi}{2}, -e^{\frac{3\pi}{2}}\right)$

d



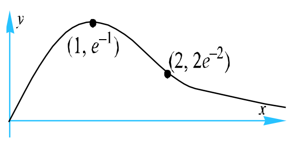
8 **a i** $e^x(\cos x - \sin x)$ **ii** $-2\sin x.e^x$ **b i** $\frac{\pi}{4}, \frac{5\pi}{4}$ **ii** 0, π, 2π

c St.pts. $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}e^{\frac{\pi}{4}}\right), \left(\frac{5\pi}{4}, -\frac{1}{\sqrt{2}}e^{\frac{5\pi}{4}}\right)$ Inf. pts. $(0, 1), (\pi, -e^{\pi}), (2\pi, e^{2\pi})$ **d**



9 **a i** $(1 - x)e^{-x}$ **ii** $(x - 2)e^{-x}$ **b i** $x = 1$ **ii** $x = 2$

c St. pt. $(1, e^{-1})$ Inf. pt. $(2, 2e^{-2})$ **d**



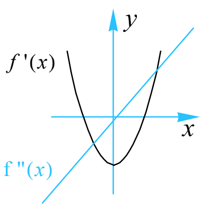
10 **a** 8 **b** 0 **c** 4 **d** $27\sqrt[3]{9} \approx 56.16$

11 **a** min value –82 **b** max value 26

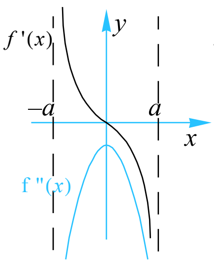
12	**a** pt A:	**i**	Yes	**ii**	non-stationary pt of inflect;
	pt B:	**i**	Yes	**ii**	Stationary point (local/global min);
	pt C:	**i**	Yes	**ii**	non-stationary pt of inflect.
	b pt A:	**i**	No	**ii**.	Local/global max;
	pt B:	**i**	No	**ii**	Local/global min;
	pt C:	**i**	Yes	**ii**	Stationary point (local max)
	c pt A:	**i**	Yes	**ii**	Stationary point (local/global max);
	pt B:	**i**	Yes	**ii**	Stationary point (local min);
	pt C	**i**	Yes	**ii**	non-stationary pt of inflect.
	d pt A:	**i**	Yes	**ii**	Stationary pt (local/global max);
	pt B:	**i**	No	**ii**	Local min;
	pt C:	**i**	Yes	**ii**	Stationary point (local max)
	e pt A:	**i**	No	**ii**	Cusp (local min);
	pt B:	**i**	Yes	**ii**	Stationary pt of inflect;
	pt C:	**i**	Yes	**ii**	Stationary point (local max)
	f pt A:	**i**	Yes	**ii**	Stationary point (local/global max);
	pt B:	**i**	Yes	**ii**	Stationary point (local/global min);
	pt C:	**i**	No	**ii**	Tangent parallel to y–axis.

13 **a i** A **ii** B **iii** C **b i** C **ii** B **iii** A

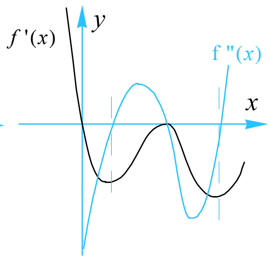
14 **a**



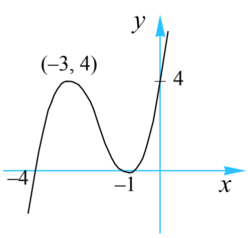
b



c



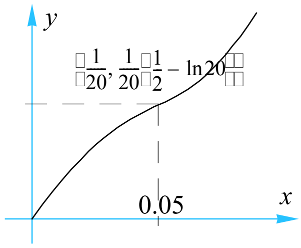
15 $y = x^3 + 6x^2 + 9x + 4$



16 $f(x) = \frac{1}{3}x^3 - x^2 - 3x - 6$

17 $f(x) = 3x^5 - 20x^3$

18



19 $m = -0.5$, $n = 1.5$

Exercise E.5.1

1. a 4 m b 2 m/s c 2 m/s^2 d 6 m/s
2. a $h'(t) = 0.5e^{0.5t}$, $h''(t) = 0.25e^{0.5t}$ b 74 m/s
3. a $h'(t)=\dfrac{4x-1}{2\sqrt{2t^2-t+1}}, 0\le t\le 5$ and $h''(t)=\dfrac{7}{4\sqrt{2t^2-t+1}\left(2t^2-t+1\right)}, 0\le t\le 5$

   b At $t = 5$ speed ≈ 1.87
4. a $d'(t)=\dfrac{-1}{(2t+1)^2}$ and $d''(t)=\dfrac{4}{(2t+1)^3}$ b -0.25 m/min
5. $v = -8$, $a = -2$

Exercise E.5.2

1 **a** $x = t^3 + 3t + 10, t \ge 0$ **b** $x = 4\sin t + 3\cos t - 1, t \ge 0$ **c** $x = t^2 - 4e^{-\frac{1}{2}t} + 2t + 4, t \ge 0$

2 **a** $x = t^3 - t^2, t \ge 0$ **b** 100 **c** $100\frac{8}{27}$ m

3 **a** $x = -\frac{2}{3}(4+t)^{3/2} + 2t + 8$ **b** 6.92 m

4 $\frac{125}{6}$ m

5 $\frac{125}{49}$ s; 63.8 m

6 **a** $\frac{\pi}{6}$ s **b** $\frac{\pi}{2} - 1$ m

7 80.37 m

8 **a** $s(t) = \frac{160}{\pi}\left[1 - \cos\left(\frac{\pi}{16}t\right)\right], t \ge 0$ **b** 86.94 m

c -6.33 m **d** 116.78 m

9 **a** $v = 4 + k - \frac{k}{t^2}, t > 0$ **b** $k = 2$ **c** 52.2 m

10 **b** 0.0893 m

Exercise E.6.1

1\. a $x^4-\frac{x^2}{2}+x+c$ b $\frac{2x^{5/2}}{5}-2x+c$ c $\frac{x^3}{3}+2x-\frac{1}{x}+c$

d $\frac{2x\sqrt{x^5}}{7}+c$ e $6\sqrt{x}+c$ f $2x^3-\frac{x^2}{2}-x+c$

2\. a $12\sqrt{x}+c$ b $6x+2\sqrt{x}+c$ c $x-\frac{2}{x}-\frac{1}{3x^3}+c$

d $\frac{3x\sqrt[3]{x}}{4}+c$ e $\frac{3x^{5/3}}{5}-\frac{3x^{4/3}}{2}+x+c$ f $\frac{-7}{5x^5}+c$

3\. a $\frac{3x^{5/3}}{5}+c$ b $\frac{3x^{7/3}}{7}+c$ c $\frac{2x\left(6x^{2/3}+15x^{1/3}+10\right)}{5}+c$

d $\frac{(x+1)^5}{5}+c$ e $\frac{(2x-1)^5}{10}+c$ f $\frac{5x^{6/5}}{6}-\frac{3x^{4/3}}{4}+c$

Exercise E.6.2

1 **a** $\frac{1}{5}e^{5x}+c$ **b** $\frac{1}{3}e^{3x}+c$ **c** $\frac{1}{2}e^{2x}+c$

d $10e^{0.1x}+c$ **e** $-\frac{1}{4}e^{-4x}+c$ **f** $-e^{-4x}+c$

g $-0.2e^{-0.5x}+c$ **h** $-2e^{1-x}+c$ **i** $5e^{x+1}+c$

j $e^{2-2x}+c$ **k** $3e^{x/3}+c$ **l** $2\sqrt{e^x}+c$

2 **a** $4\log_e x+c, x>0$ **b** $-3\log_e x+c, x>0$ **c** $\frac{2}{5}\log_e x+c, x>0$

d $\log_e(x+1)+c, x>-1$ **e** $\frac{1}{2}\log_e x+c, x>0$ **f** $x-2\log_e x-\frac{1}{x}+c, x>0$

g $\frac{1}{2}x^2-2x+\log_e x+c, x>0$ **h** $3\ln(x+2)+c$

3 **a** $-\frac{1}{3}\cos(3x)+c$ **b** $\frac{1}{2}\sin(2x)+c$ **c** $\frac{1}{5}\tan(5x)+c$ **d** $\cos(x)+c$

4 **a** $-\frac{1}{2}\cos(2x)+\frac{1}{2}x^2+c$ **b** $2x^3-\frac{1}{4}\sin(4x)+c$ **c** $\frac{1}{5}e^{5x}+c$

d $-\frac{4}{3}e^{-3x}-2\cos\left(\frac{1}{2}x\right)+c$ e $3\sin\left(\frac{x}{3}\right)+\frac{1}{3}\cos(3x)+c$

f $\frac{1}{2}e^{2x}+4\log_e x-x+c, x>0$ g $\frac{1}{2}e^{2x}+2e^{x}+x+c$

h $\frac{5}{4}\cos(4x)+x-\log_e x+c, x>0$ i $\frac{1}{3}\tan(3x)-2\log_e x+2e^{x/2}+c, x>0$

j $\frac{1}{2}e^{2x}-2x-\frac{1}{2}e^{-2x}+c$ k $\frac{1}{2}e^{2x+3}+c$

l $-\frac{1}{2}\cos(2x+\pi)+c$ m $\sin(x-\pi)+c$

n $-4\cos\left(\frac{1}{4}x+\frac{\pi}{2}\right)+c$ o $2\left(\frac{e^x+2}{\sqrt{e^x}}\right)+c$

5 a $\frac{1}{16}(4x-1)^4+c$ b $\frac{1}{21}(3x+5)^7+c$ c $-\frac{1}{5}(2-x)^5+c$

d $\frac{1}{12}(2x+3)^6+c$ e $-\frac{1}{27}(7-3x)^9+c$ f $\frac{1}{5}\left(\frac{1}{2}x-2\right)^{10}+c$

g $-\frac{1}{25}(5x+2)^{-5}+c$ h $\frac{1}{4}(9-4x)^{-1}+c$ i $-\frac{1}{2}(x+3)^{-2}+c$

j $\ln(x+1)+c, x>-1$ k $\ln(2x+1)+c, x>-\frac{1}{2}$ l $-2\ln(3-2x)+c, x<\frac{3}{2}$

m $3\ln(5-x)+c, x<5$ n $-\frac{3}{2}\ln(3-6x)+c, x<\frac{1}{2}$ o $\frac{5}{3}\ln(3x+2)+c, x>-\frac{2}{3}$

6 a $-\frac{1}{2}\cos(2x-3)-x^2+c$ b $6\sin\left(2+\frac{1}{2}x\right)+5x+c$ c $\frac{3}{2}\sin\left(\frac{1}{3}x-2\right)+\ln(2x+1)+c$

d $10\tan(0.1x-5)-2x+c$ e $2\ln(2x+3)+2e^{-\frac{1}{2}x+2}+c$ f $-\frac{2}{2x+3}-\frac{1}{2}e^{2x-\frac{1}{2}}+c$

g $x+\ln(x+1)-4\ln(x+2)+c$ h $2x-3\ln(x+2)+\frac{1}{2}\ln(2x+1)+c$

i $-\frac{1}{2x+1}+\ln(2x+1)+c$

7 a $f(x) = \frac{1}{6}\sqrt{(4x+5)^3}$ b $f(x) = 2\ln(4x-3)+2$

c $f(x) = \frac{1}{2}\sin(2x+3)+1$ d $f(x) = 2x+\frac{1}{2}e^{-2x+1}+\frac{1}{2}e$

8 14 334

9 13.19ms^{-1} or 1.19ms^{-1}

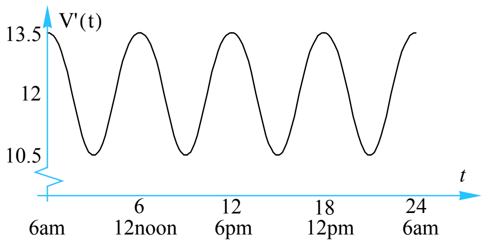
10 2.66 cm

11 $2e^{x/2}-\frac{1}{2}\sin(2x)-2$

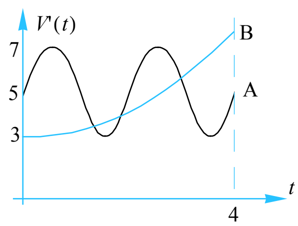
12 a $p = \frac{a}{a^2+b^2}$, $q = -\frac{b}{a^2+b^2}$ b $\frac{1}{13}e^{2x}(2\sin 3x-3\cos 3x)+c$

13 a $0.25a$ b $a\times\left(\frac{1}{2}\right)^{8/3}\approx 0.1575a$

14 **b** 666 g

15 a 

b 73.23% **c** ~25.24 litres

16 a 

b 7000 **c** 1.16 day **d** 2 days

Exercise E.6.3

1 a $\frac{2}{3}(x^2+1)^{3/2}+c$ b $\frac{2}{3}(x^3+1)^{3/2}+c$ c $-\frac{1}{3}(4-x^4)^{1.5}+c$

d $\ln(x^3+1)+c$ e $-\frac{1}{18(3x^2+9)^3}+c$ f $e^{(x^2+4)}+c$

g $\ln(z^2+4z-5)+c$ h $-\frac{3}{8}(2-t^2)^{4/3}+c$ i $e^{\sin x}+c$

	j	$\ln[e^x+1]+c$	k	$\frac{1}{5}\sin^5 x+c$	l	$\frac{2}{5}(x+1)^{5/2}-\frac{2}{3}(x+1)^{3/2}+c$
2	a	$\frac{1}{10}(2x-1)^{5/2}+\frac{1}{6}(2x-1)^{3/2}+c$	b	$-\frac{2}{3}(1-x)^{3/2}+\frac{4}{5}(1-x)^{5/2}-\frac{2}{7}(1-x)^{7/2}+c$		
	c	$\frac{2}{5}(x-1)^{5/2}+\frac{4}{3}(x-1)^{3/2}+c$	d	$e^{\tan x}+c$	e	$-\ln(1-2x^2)+c$
	f`	$\frac{1}{1-2x^2}+c$	g	$\frac{1}{2}(\ln x)^2+c$	h	$-\ln(1+e^{-x})+c$
	i	$\ln(\ln x)+c$				
3	a	0	b	$\frac{2\ln 2}{3}$	c	$\ln\frac{77}{54}$
	d	$\ln 2$	e	$\frac{1}{3}\ln 2$	f	$\frac{1}{4}$
	g	$\frac{76}{15}$	h	$\frac{16}{15}$	i	$\frac{2}{3}(1+e)^{3/2}(1-e^{-3/2})$
4	a	$\frac{7\sqrt{7}}{3}-\frac{8}{3}$	b	$\frac{3}{8}(\cos\pi^2-1)$	c	$\frac{1042}{5}$
	d	$\ln 4$	e	1	f	$\frac{5}{4}(e^5-e^{-1})$
	g	24 414	h	$\sqrt{3}-\sqrt{2}$	i	$\frac{1}{4}\ln 3$
5	a	$\frac{1}{4}$	b	$2-\frac{2}{3}\sqrt{3}$	c	$\frac{31}{80}$
	d	$4-2\sqrt{2}$	e	$\ln 2$	f	$\frac{2}{3}$
6	a	$-\frac{2}{5}\sqrt{3}$	b	$\frac{2}{5}\sqrt{3}$	c	$\frac{26}{3}$
	d	$-\frac{4}{3}$	e	$-\frac{56}{15}\sqrt{2}$	f	3 + 2ln4

7 **a** $\text{Tan}^{-1}(x+3)+c$ **b** $\frac{2}{\sqrt{3}}\text{Tan}^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+c$ **c** $\text{Sin}^{-1}\left(\frac{x-2}{\sqrt{5}}\right)+c$

d $3\,\text{Sin}^{-1}\left(\frac{x+1}{3}\right)+c$ **e** $2\sin^{-1}\left(\frac{2x-3}{\sqrt{29}}\right)+c$ **f** $\frac{1}{2}\sin^{-1}\left(\frac{x^2}{3}\right)+c$

g $\frac{1}{2}(\arcsin x)^2+c$ **h** $-\frac{1}{3}(\arccos x)^3+c$ **i** $-\frac{1}{2}(\arcsin x)^{-2}+c$

8 **a** $A = 1, B = -2$

9 **a** $\text{Tan}^{-1}k$ **b i** $\frac{\pi}{6}$ **ii** $\frac{\pi}{4}$ **c** $\frac{\pi}{2}, \pi$

10 $2\sqrt{x}-2\ln(\sqrt{x}+1), 2-2\ln 2$

11 $\frac{3k^2\pi}{8}$

12 $\frac{\pi a^2}{4}$

13 **a** $\frac{\pi}{3}$ **b** $8\,\text{Sin}^{-1}\left(\frac{2}{3}\right)$ **c** $\frac{\pi}{4}$

d $\frac{1}{2}\text{Sin}^{-1}(1)$ **e** $2\sqrt{2}-2-\frac{\pi}{2}$ **f** $\frac{\pi}{4}$

g $\pi-2\,\text{Tan}^{-1}\left(\frac{1}{3}\right)$

Exercise E.6.4

1 **a** x^2+x+3 **b** $2x-\frac{1}{3}x^3+1$ **c** $\frac{8}{3}\sqrt{x^3}-\frac{1}{2}x^2-\frac{40}{3}$

d $\frac{1}{2}x^2+\frac{1}{x}+2x-\frac{3}{2}$ **e** $(x+2)^3$ **f** $\frac{3}{4}\sqrt[3]{x^4}+\frac{1}{4}x^4+x$

g $\frac{1}{3}x^3+1$ **h** x^4-x^3+2x+3

2 $\frac{1}{2}x^2+\frac{1}{x}+\frac{5}{2}$

3 $3835.03

4 9.5

5 $\frac{251}{3}\pi$ cm³

6 292

7 $\frac{5}{7}\sqrt{x^3} + \frac{23}{7}$

8 1, –8

9 $P(x) = 25 - 5x + \frac{1}{3}x^2$

10 $N = \frac{20000}{201}t^{2.01} + 500, t \geq 0$

11 **a** $y = -\frac{2}{5}x^2 + 4x$ **b** $y = \frac{1}{6}x^3 + \frac{5}{4}x^2 + 2x$

12 $y = 2(x^3 + x^2 + x)$

13 $f(x) = -\frac{3}{10}x^3 + \frac{49}{10}x - \frac{13}{5}$

14 Vol ~ 43 202 cm³

15 110 cm²

Exercise E.6.5

1 **a** $\frac{15}{2}$ **b** $\frac{38}{3}$ **c** $\frac{5}{36}$ **d** –8

2 **a** $\frac{35}{24}$ **b** $\frac{8}{5}\sqrt{2} - 2$ **c** –2 **d** 0

e $\frac{1}{20}$ **f** $-\frac{4}{3}$ **g** $\frac{7}{6}$ **h** $\frac{5}{6}$

i $\frac{20}{3}$ **j** 0 **k** $\frac{20}{3}$ **l** $-\frac{\sqrt{2}}{3}$

4 **a** e **b** $2(e^{-2} - e^{-4})$ **c** 0 **d** $2(e - e^{-1})$

e $e^2 + 4 - e^{-2}$ **f** $\frac{1}{2}(e - e^5)$ **g** $2\sqrt{e} - 3$

h $\frac{1}{4}(16e^{1/4} - e^4 - 15)$ **i** $\frac{1}{2}(e^{-1} - e^3)$

6 **a** $3\ln 2$ **b** 2ln5 **c** 4 + 4ln3 **d** $\frac{1717}{4}$

e $\frac{3}{2}\ln 3$ **f** 2ln2 **g** $\frac{3}{4}$ **h** 4ln2 – 2 **i** ln2

8 **a** 1 **b** $\frac{3\sqrt{3}}{2}$ **c** $\frac{\sqrt{3}}{2}$ **d** -2

e $\frac{\pi^2}{32}-1$ **f** 0 **g** 0 **h** $\frac{\sqrt{3}}{2}-\frac{1}{2}$

i 0 **j** 2

9 **a** $\frac{31}{5}$ **b** $\frac{7\sqrt{7}}{3}-\sqrt{3}$ **c** 0 **d** $\frac{5}{72}$

e $3\sqrt[3]{2}-\frac{3}{2}$ **f** $1-\ln 2$ **g** $\frac{76}{15}$ **h** $\frac{16}{15}$

i $\frac{2}{3}(e+1)^{3/2}(1-e^{-3/2})$

10 $\ln\left(\frac{21}{5}\right)$

11 $\sin 2x + 2x\cos 2x$; 0

12 **a** $2m-n$ **b** $m+a-b$ **c** $-3n$ **d** $m(2a-b)$ **e** na^2

13 **a** $e^{0.1x}+0.1xe^{0.1x}$; $10xe^{0.1x}-100e^{0.1x}+c$

b i 99 accidents **ii** $N = 12t+10te^{0.1t}-100e^{0.1t}+978$

14 **a** 1612 subscribers **b** 46 220

15 **b** ~524 flies

Exercise E.6.6

1 **a** 4 sq.units **b** $\frac{32}{3}$ sq.units **c** 4 sq.units

d 36 sq.units **e** $\frac{1}{6}$ sq.units

2 **a** e sq.units **b** $\frac{1}{2}(e^4-2-e^2)$ sq.units **c** $2(e+e^{-1}-2)$ sq.units

d $2(e^2-2-e)$ sq.units

3 **a** $\ln\left(\frac{5}{4}\right)$ sq.units **b** 2ln5 sq.units **c** 3ln3 sq.units **d** 0.5 sq.units

4 **a** 2 sq.units **b** $\frac{\pi}{2}$ sq.units **c** $\frac{3}{8}\pi^2+\sqrt{2}-2$ sq.units

d $\sqrt{2}$ sq. units **e** $4\sqrt{3}$ sq.units

6 12 sq. units

7 $4\left(\sqrt{3}-\frac{1}{3}\right)$ sq.units.

8 ln2 + 1.5 sq.units.

9 2 sq.units.

10 $\frac{37}{12}$ sq. units

11 **a** 0.5 sq. units **b** 1 sq. unit **c** $2(\sqrt{6}-\sqrt{2})$ sq. units

12 $\frac{8}{3}$

13 $-2\tan 2x$; $\frac{1}{4}\ln 2$ sq.units

14 **a** $\frac{9}{2}$ sq. units **b** 3 sq. units

15 **a** 1 sq.unit **b** 10 sq. units

16 **a** $x\ln x - x + c$ **b** 1 sq. unit

17 $\frac{14}{3}$ sq. units

18 **a** $\frac{7}{6}$ sq. units **b** $\frac{9}{2}$ sq. units

19 **a i** $\frac{15}{4}$ sq. units **ii** $\frac{45}{4}$ sq. units

20 $\frac{22}{3}$ sq. units

21 **b i** $e^{-1}+e-2$ sq. units **ii** 1 sq. unit **iii** 2ln(2) sq. units

22 **b** 3.05 sq. units

23 **a** $2y = 3ax - a^3$ **b** $\frac{1}{15}a^5$ sq. units

24 **a** $1-e^{-1}$ sq. units **b** e^{-1} sq. units **c** $1-e^{-e^{-1}-1}-e^{-1}$ ~ 0.10066 sq. units

Exercise E.11.1

1. 5.6 2. 3.423 3. −1.782 4. 5.36
5. 4.04 6. 0.648 7. 3.144 8. ~ 2.2991×10^{29}
9. ~ 7.190 10. ~ 0.736 11. ~ 2.629 12. ~4.240
13. ~ 5.084

Exercise E.11.2

1. a $y=\frac{x^2}{2}+x+2$ b $y=\sqrt{x^2+2x+1}$ c $\frac{y^2}{2}-y-4=\frac{-\left(x^2=2x-3\right)}{2}$
d $y=\sqrt{e^{-2x}\left((2x-1)e^{2x}+1\right)}$ e $y=0$
f $r=\frac{4s^{3/2}}{3}+2\sqrt{s}$ g $y=-\frac{1}{\sqrt{3-2\sqrt{1+x^2}}}$ h $A=\sqrt{\frac{8r^3}{3}}$
i $y\log_e y-y=\frac{x^2}{2}$ j $y^{3/2}=\frac{3x(x-6)}{4}$

2. a $\frac{dC}{dt}=0.01\sqrt{t}, C(0)=0$ b $C=\frac{2}{300}t\sqrt{t}$ c 2.357

3. a $\frac{dC}{dt}=kC$ b $C=e^{kt}$ c $\frac{\log_e 2}{k}$

4. a $\frac{dP}{dt}=\frac{5.98-5.23}{5}=0.15$ b No.

5. a $\frac{dV}{dt}=k$ b $V=kt+c$

6. a $e^{-y}\cos y\,dy=e^{-t}\left(1+t^2\right)dt$ c $\frac{e^{-y}}{2}(\sin y-\cos y)=-e^{-t}\left(t^2+2t+3\right)+c$ d $\frac{5}{2}$

Exercise E.11.3

1. a $kx=e^{y/x}$ b $kx=e^{-x/y}$ c $y^2=kx^3+x^2$ d $\log_e(kx)=-e^{-y/x}$
e $x=kx^2(x-2y)$ f $x^2-2xy-y^2=k$

2. $y^2+x^2=5x^3$ 3. $x^2=y\left(1+2\log_e y\right)$

8. $y=x-x\cos x$ 9. $y^2=-2x^2\log_e x$ 10. $y=x(2x-1)$

Exercise E.11.4

1 (a) $\frac{dy}{dx} - \frac{2}{x}y = x^4, \quad x > 0.$

$I(x) = e^{\int -\frac{2}{x}dx} = e^{-2\ln(x)} = \frac{1}{x^2}.$

So $\frac{1}{x^2}\cdot\frac{dy}{dx} - \frac{2}{x^3}y = x^2$

$\Rightarrow \frac{d}{dx}\left(\frac{1}{x^2}\cdot y\right) = x^2$

Integrate b.s.w.r.t. x: $\frac{y}{x^2} = \frac{x^3}{3} + c$

$\Rightarrow y = \frac{x^5}{3} + cx^2.$

(b) $\frac{dy}{dx} + \frac{2}{x}y = x^4, \quad x > 0.$

$I(x) = e^{\int \frac{2}{x}dx} = e^{2\ln(x)} = x^2.$

So $x^2\frac{dy}{dx} + 2xy = x^6$

$\Rightarrow \frac{d}{dx}\left(x^2y\right) = x^6$

Integrate b.s.w.r.t. x: $x^2y = \frac{x^7}{7} + c$

$\Rightarrow y = \frac{x^5}{7} + cx^{-2}.$

(c) $\frac{dy}{dx} + 2y = e^{-x}$, $\quad I(x) = e^{\int 2dx} = e^{2x}$.

So: $e^{2x}\frac{dy}{dx} + 2e^{2x}y = e^{x}$

$\Rightarrow \frac{d}{dx}\left(e^{2x}y\right) = e^{x}$

Integrate b.s.w.r.t. x: $e^{2x}y = e^{x} + c$

$\Rightarrow \underline{y = e^{-x} + ce^{-2x}}$.

(d) $\frac{dy}{dx} + 2y = x$, $\quad I(x) = e^{\int 2dx} = e^{2x}$.

So: $e^{2x}\frac{dy}{dx} + 2e^{2x}y = xe^{2x}$

$\Rightarrow \frac{d}{dx}\left(e^{2x}y\right) = xe^{2x}$

Integrate b.s.w.r.t. x: $e^{2x}y = \int xe^{2x}\,dx$

$\Rightarrow e^{2x}y = \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\,dx$

$\Rightarrow e^{2x}y = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$

$\Rightarrow \underline{y = \frac{x}{2} - \frac{1}{4} + ce^{-2x}}$.

(e) $\frac{dy}{dx} + \frac{1}{x}y = \frac{\ln(x)}{x^2}, \quad x > 0.$

$$I(x) = e^{\int \frac{1}{x} dx} = e^{\ln(x)} = x.$$

So: $x\frac{dy}{dx} + y = \frac{\ln(x)}{x^2}$

$$\Rightarrow \frac{d}{dx}(xy) = \frac{\ln(x)}{x^2}$$

Integrate b.s.w.r.t. x:

$$xy = \int \frac{1}{x^2}\ln(x)\,dx = -\frac{1}{x}\ln(x) + \int \frac{1}{x}\cdot\frac{1}{x}\,dx$$

$$= -\frac{1}{x}\ln(x) - \frac{1}{x} + c$$

$$\Rightarrow y = -\frac{1}{x^2}\ln(x) - \frac{1}{x^2} + \frac{c}{x}.$$

(f) $\frac{dy}{dx} + \frac{y}{x-1} = x,$ $\quad I(x) = e^{\int \frac{1}{x-1}dx} = e^{\ln(x-1)} = x-1.$

So, $(x-1)\frac{dy}{dx} + y = x(x-1)$

Integrate b.s.w.r.t. x:

$$y(x-1) = \int (x^2 - x)\,dx = \frac{x^3}{3} - \frac{x^2}{2} + c$$

$$\Rightarrow y = \frac{x^3}{3(x-1)} - \frac{x^2}{2(x-1)} + \frac{c}{x-1}$$

$$y = \frac{2x^3 - 3x^2 + 6c}{6(x-1)}$$

② (a) $\frac{dy}{dx} = e^{-x} - y, \quad y(0) = 1.$

$\Rightarrow \frac{dy}{dx} + y = e^{-x}$ and $I(x) = e^{\int 1 dx} = e^{x}.$

$\Rightarrow e^{x}\frac{dy}{dx} + e^{x}y = 1$

$\frac{d}{dx}\left(e^{x}y\right) = 1$

Integrate b.s.w.r.t. x: $e^{x}y = x + c$

and $y(0) = 1$, so $c = 1$.

Then $\underline{y = xe^{-x} + e^{-x}}$.

(b) $(x^2+1)\frac{dy}{dx} - xy = 0, \quad y(0) = 3.$

$\Rightarrow \frac{dy}{dx} - \frac{x}{x^2+1}\cdot y = 0$ and $I(x) = e^{-\int \frac{x}{x^2+1} dx}$

$= e^{-\frac{1}{2}\ln(x^2+1)}$

$= \frac{1}{\sqrt{x^2+1}}$

$\Rightarrow \frac{d}{dx}\left(\frac{1}{\sqrt{x^2+1}}\, y\right) = 0$

Integrate b.s.w.r.t. x: $\frac{y}{\sqrt{x^2+1}} = c$

and $y(0) = 3 \Rightarrow c = 3$

The solution is: $\underline{y = 3\sqrt{x^2+1}}$.

(c) $5\dfrac{dx}{dt} + \dfrac{15x}{50-t} = 1, \qquad x(0) = -45.$

$$\Rightarrow \frac{dx}{dt} + \frac{3}{50-t}x = \frac{1}{5}$$

and $I(t) = e^{\int \frac{3}{50-t}dt} = e^{-3\ln(50-t)} = (50-t)^{-3}.$

$$\text{So } (50-t)^{-3}\frac{dx}{dt} + 3(50-t)^{-4}x = \frac{1}{5}(50-t)^{-3}$$

$$\Rightarrow \frac{d}{dx}\left((50-t)^{-3}x\right) = \frac{1}{5}(50-t)^{-3}$$

Integrate b.s.w.r.t. x: $\quad (50-t)^{-3}x = -\frac{1}{10}(50-t)^{-4} + c$

and $x(0) = -45$, so,

$$c = (50)^{-3}(-45) + \frac{1}{10}(50)^{-4} = -0.0036.$$

$$\text{So} \quad x = -\frac{1}{10(50-t)} - \frac{0.0036}{(50-t)^3}.$$

(d) $x\dfrac{dy}{dx} + y = 4x^2, \qquad y(1) = 0.$

$\Rightarrow \dfrac{dy}{dx} + \dfrac{1}{x}y = 4x$ and $I(x) = e^{\int \frac{1}{x}dx} = x.$

$$\text{So} \quad x\frac{dy}{dx} + y = 4x^2$$

$$\Rightarrow \frac{d}{dx}(xy) = 4x^2$$

Integrate w.r.t. x: $\quad xy = \frac{4}{3}x^3 + c$

and $y(1) = 0$, so $c = -\frac{4}{3}.$

$$\text{So} \quad y = \frac{4}{3}x^2 - \frac{4}{3x}.$$

(e) $\frac{dy}{dx} = \frac{1}{x+y^2}$, $y(-1)=0$

$\Rightarrow \frac{dx}{dy} = x + y^2 \Rightarrow \frac{dx}{dy} - x = y^2$

Now $I(y) = e^{\int -1\,dy} = e^{-y}$

and $e^{-y}\frac{dx}{dy} - e^{-y}x = y^2 e^{-y}$

$\Rightarrow \frac{d}{dy}\left(e^{-y}x\right) = y^2 e^{-y}$

Integrate b.s. w.r.to y: $e^{-y}x = \int y^2 e^{-y}\,dy$

Integration by parts twice gives

R.H.S $= -y^2e^{-y} - 2ye^{-y} - e^{-y} + c$

So $x = -y^2 - 2y - 1 + ce^{-y}$

and $y(-1)=0$, so $-1 = -1 + c \Rightarrow c = 0$.

We have $\underline{x = -y^2 - 2y - 1}$.

(f) $\frac{dy}{dx} + \frac{2}{x}y = \frac{\sin(x)}{x^2}$, $y(\pi)=1$, $x>0$.

$I(x) = e^{\int \frac{2}{x}dx} = e^{2\ln(x)} = x^2$.

So $x^2\frac{dy}{dx} + 2xy = \sin(x)$

$$\Rightarrow \frac{d}{dx}\left(x^2 y\right) = \sin(x)$$

Integrate b.s.w.r.t. x : $x^2 y = -\cos(x) + c$

and $y(\pi) = 1$, so $c = \pi^2 - 1$.

We have $x^2 y = -\cos(x) + \pi^2 - 1$

or $y = \frac{\pi^2 - 1}{x^2} - \frac{\cos(x)}{x^2}$.

(3) $y' \cos(x) = y \sin(x) + \sin(2x)$

$$\Rightarrow y' \cos(x) - \sin(x) \cdot y = \sin(2x)$$

We recognise this is in the form

$$y' I(x) + I'(x) y = \sin(2x)$$

$$\Rightarrow \frac{d}{dx}\left(\cos(x) \cdot y\right) = \sin(2x)$$

Integrate b.s.w.r.t. x : $\cos(x) y = -\frac{1}{2}\cos(2x) + c$

and so: $2\cos(x) y = 2c - \cos(2x)$.

(4) $(1+x)\frac{dy}{dx} + y = 1 + x$, $y(1) = \frac{1}{4}$.

$$\Rightarrow \frac{dy}{dx} + \frac{1}{1+x} y = 1$$

and $I(x) = e^{\int \frac{1}{1+x} dx} = e^{\ln(1+x)} = 1 + x$.

$$\Rightarrow \frac{d}{dx}\left((1+x) y\right) = 1 + x$$

Integrate b.s. w.r.t. x: $(1+x)y = x + \frac{x^2}{2} + c$

and $y(1) = \frac{1}{4}$, so $c = 2.\frac{1}{4} - 1 - \frac{1}{2} = -1$.

Then we have: $\underline{(1+x)y = x + \frac{x^2}{2} - 1}$.

(5) $\frac{dy}{dx} + y = p(x)$ where $p(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$

$I(x) = e^{\int 1\,dx} = e^x$, so $\frac{d}{dx}(e^x y) = e^x p(x)$

$\Rightarrow e^x y = \int e^x p(x)\,dx$.

For $0 \le x \le 1$, $p(x) = 1$ giving $e^x y = e^x + c_1$.

For $x > 1$, $p(x) = 0$ giving $e^x y = c_2$.

Therefore, $\underline{y = \begin{cases} 1 + c_1 e^{-x}, & 0 \le x \le 1 \\ c_2 e^{-x}, & x > 1 \end{cases}}$

(6) $\frac{dy}{dx} + \frac{xy}{4-x^2} = 1$, $y(0) = 1$, $|x| < 2$.

So $I(x) = e^{\int \frac{x}{4-x^2}dx} = e^{-\frac{1}{2}\ln(4-x^2)} = (4-x^2)^{-\frac{1}{2}}$.

$\Rightarrow (4-x^2)^{-1/2}\frac{dy}{dx} + \frac{x}{(4-x^2)^{3/2}}y = (4-x^2)^{-1/2}$

$\Rightarrow \frac{d}{dx}\left((4-x^2)^{-1/2}y\right) = \frac{1}{\sqrt{4-x^2}}$

$\Rightarrow \frac{1}{\sqrt{4-x^2}}y = \int \frac{1}{\sqrt{4-x^2}}dx = \arcsin\left(\frac{x}{2}\right) + c$

and $y(0)=1$, so $c=\frac{1}{2}$

Therefore, $\frac{1}{\sqrt{4-x^2}}\,y = \arcsin\left(\frac{x}{2}\right)+\frac{1}{2}$

or $\underline{y = \sqrt{4-x^2}\,\arcsin\left(\frac{x}{2}\right)+\frac{1}{2}\sqrt{4-x^2}}$.

(7) $u = x^2 \Rightarrow \frac{du}{dy} = 2x\frac{dx}{dy}$ —(1)

Now, given $\frac{dy}{dx} = \frac{x}{x^2y+y^3} \Rightarrow \frac{dx}{dy} = xy+\frac{y^3}{x}$ —(2)

From (1) & (2): $\frac{1}{2x}\frac{du}{dy} = xy+\frac{y^3}{x}$

$\Rightarrow \frac{du}{dy} = 2x^2y+2y^3 = 2uy+2y^3$.

So $\underline{\frac{du}{dy} - 2yu = 2y^3}$.

Now $I(y) = e^{\int -2y\,dy} = e^{-y^2}$.

$\Rightarrow e^{-y^2}\frac{du}{dy} - 2ye^{-y^2}u = 2y^3e^{-y^2}$

$\Rightarrow \frac{d}{dy}\left(e^{-y^2}u\right) = 2y^3e^{-y^2}$

Integrate b.s.w.r.t. y:

$e^{-y^2}u = 2\int y^3e^{-y^2}\,dy$

RHS cannot be integrated simply, so

$$x^2 e^{-y^2} = 2\int y^3 e^{-y^2}\, dy.$$

(8) $\tan(x)\dfrac{dy}{dx} + y = \sin(x), \quad y(\pi) = 0.$

$\Rightarrow \dfrac{dy}{dx} + \dfrac{\cos(x)}{\sin(x)} y = \cos(x)$

and $I(x) = e^{\int \frac{\cos(x)}{\sin(x)} dx} = e^{\ln(\sin(x))} = \sin(x).$

So $\sin(x)\dfrac{dy}{dx} + \cos(x) y = \sin(x)\cos(x)$

$\Rightarrow \dfrac{d}{dx}\left(\sin(x) y\right) = \frac{1}{2}\sin(2x)$

Integrate b.s. w.r.t. x: $\sin(x) y = -\frac{1}{4}\cos(2x) + c.$

And $y(\pi) = 0$, so $c = 0 - \frac{1}{4} = -\frac{1}{4}.$

The solution is $\sin(x) y = -\frac{1}{4}\cos(2x) - \frac{1}{4}.$

(9) (a) No change in volume of solution in tank.

Let $s(t)$ be the amount of salt in the tank at time, t mins.

Let a = amount of salt in (in kg/l)

Amount of salt in $= a$ kg/l $\times$ 10 l/m $= 10a$ kg/m

Amount of salt out $= \dfrac{s}{100}$ kg/l $\times$ 10 l/m $= \dfrac{10s}{100}$ kg/m

So $\dfrac{ds}{dt} = 10a - \dfrac{s}{10}$ (kg/min)

(i) $a = 0$, so $\dfrac{ds}{dt} = -\dfrac{s}{10}$

$$\Rightarrow \int \frac{10}{s}\frac{ds}{dt}\,dt = -\int dt$$

$$\Rightarrow 10\ln(s) = -t + c$$

At $t=0$, $s=80$, so $c = 10\ln(80)$

$$\Rightarrow 10\ln(s) = 10\ln(80) - t$$

$$\Rightarrow \ln(s) = \ln(80) - t/10$$

$$\Rightarrow s = e^{-t/10} \cdot e^{\ln(80)}$$

$$\Rightarrow \underline{s = 80e^{-t/10}}.$$

(ii) $a=1$, so $\frac{ds}{dt} = 10 - \frac{s}{10} = \frac{100-s}{10}$

$$\Rightarrow \int \frac{10}{100-s}\frac{ds}{dt}\,dt = \int dt$$

$$\Rightarrow -10\ln(100-s) = t + c$$

At $t=0$, $s=80$, so $c = -10\ln(20)$

$$\Rightarrow -10\ln(100-s) = t - 10\ln(20)$$

$$\Rightarrow \ln(100-s) = -t/10 + \ln(20)$$

$$\Rightarrow 100 - s = e^{-t/10} \cdot e^{\ln(20)}$$

$$= 20e^{-t/10}$$

$$\Rightarrow \underline{s = 100 - 20e^{-t/10}}.$$

9 (b) Amount of salt in = 10 a kg/min

Amount of salt out = $\frac{10s}{100+2t}$ kg/min

So $\frac{ds}{dt} = 10a - \frac{10s}{100+2t}$

(i) $a = 0$: $\frac{ds}{dt} = -\frac{10s}{100+2t}$

$\Rightarrow \int \frac{1}{s}\frac{ds}{dt}dt = -10\int \frac{1}{100+2t}dt$

$\Rightarrow \ln(s) = -5\ln(100+2t) + c$

$t=0,\ s=80$, so $c = \ln(80) + 5\ln(100)$

So $\ln(s) = -5\ln(100+2t) + \ln 80 + 5\ln(100)$

$\Rightarrow s = e^{-5\ln(100+2t)} \cdot e^{\ln 80} \cdot e^{+5\ln(100)}$

$= (100+2t)^{-5} \cdot 80 \cdot 100^{+5}$

$\Rightarrow \underline{s = 80\left(\frac{100}{100+2t}\right)^5}$

(ii) $a = 1$: $\frac{ds}{dt} = 10 - \frac{10s}{100+2t}$

$\Rightarrow \frac{ds}{dt} + \frac{10}{100+2t}s = 10$

So $P(t) = \frac{10}{100+2t}$ and $I(t) = e^{\int \frac{10}{100+2t}dt}$

$\Rightarrow I(t) = e^{5\ln(100+2t)} = (100+2t)^5$.

$$\Rightarrow (100+2t)^5 \frac{ds}{dt} + 10(100+2t)^4 s = 10(100+2t)^5$$

$$\Rightarrow \frac{d}{dt}\left((100+2t)^5 s\right) = 10(100+2t)^5$$

$$\Rightarrow \int \frac{d}{dt}\left((100+2t)^5 s\right) dt = 10\int (100+2t)^5 dt$$

$$\Rightarrow (100+2t)^5 s = \frac{10}{12}(100+2t)^6 + c$$

and at $t=0$, $s=80$ so

$$c = 100^5 \cdot 80 - \frac{5}{6} \cdot 100^6 = -\frac{10}{3} \cdot 100^5.$$

We have: $s = \frac{5}{6}(100+2t) - \frac{10}{3}\left(\frac{100}{100+2t}\right)^5.$

(10) $R\frac{dQ}{dt} + \frac{Q}{C} = V$, $Q(0) = 0$.

$$\Rightarrow \frac{dQ}{dt} + \frac{1}{RC}Q = \frac{V}{R}, \quad R, C, V \text{ constants.}$$

So $I(t) = e^{\int \frac{1}{RC}dt} = e^{t/RC}$.

So $e^{t/RC}\frac{dQ}{dt} + e^{t/RC} \cdot \frac{1}{RC} \cdot Q = \frac{V}{R}e^{t/RC}$

$$\Rightarrow \frac{d}{dt}\left(e^{t/RC}Q\right) = \frac{V}{R}e^{t/RC}$$

Integrate b.s.w.r.t. t: $e^{t/RC}Q = CVe^{t/RC} + c$

And $Q(0) = 0$, so $c = -CV$

$$\Rightarrow e^{t/RC}Q = CVe^{t/RC} - CV \Rightarrow Q = CV\left(1 - e^{-t/RC}\right)$$

(11) (a) $\frac{1}{2}\frac{di}{dt} + 10i = 10, \quad i(0) = 0$

So $I(t) = e^{\int 20\,dt} = e^{20t}$.

$$e^{20t}\frac{di}{dt} + 20e^{20t}i = 20$$

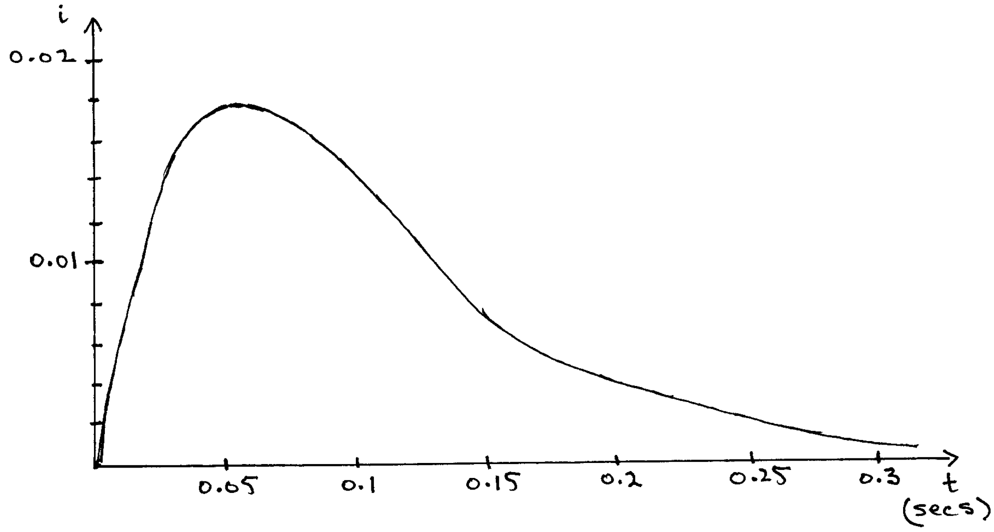
$$\Rightarrow \frac{d}{dt}\left(e^{20t}i\right) = 20$$

Integrate b.s.w.r.t. t: $20e^{20t}i = 20t + c$.

And $i(0) = 0$, so $c = 0$

Solution is $\underline{i = te^{-20t}}$.

(b)



(c) (i) $t = 0.05$, $i = 0.05e^{-1} \approx \underline{0.018}$.

(ii) Maximum occurs when $\frac{di}{dt} = 0$.

$$\frac{di}{dt} = e^{-20t} - 20te^{-20t}$$

$$= e^{-20t}(1 - 20t) = 0 \text{ when } t = \frac{1}{20}$$

Since $e^{-20t} > 0$ for all t.

$\therefore$ Maximum current is when $t = 0.05$,

that is $\underline{\approx 0.018}$ (see (c)(i)).

(12) $\frac{dx}{dt} + \alpha x = \beta(t)$

When $\beta(t) = \beta$ and $\beta \in \mathbb{R}$ we have

$$\frac{dx}{dt} + \alpha x = \beta$$

$$I(t) = e^{\int \alpha \, dt} = e^{\alpha t}.$$

$$\text{So} \quad \frac{d}{dt}\left(e^{\alpha t} x\right) = \beta e^{\alpha t}$$

Integrate b.s.w.r.t. t: $e^{\alpha t} x = \frac{\beta}{\alpha} e^{\alpha t} + c$

$$\text{So} \quad x = \frac{\beta}{\alpha} + c e^{-\alpha t}.$$

Logically, when $t = 0$, $x = 0$, so

$$c = -\frac{\beta}{\alpha}$$

$$\text{So} \quad x = \frac{\beta}{\alpha} - \frac{\beta}{\alpha} e^{-\alpha t}$$

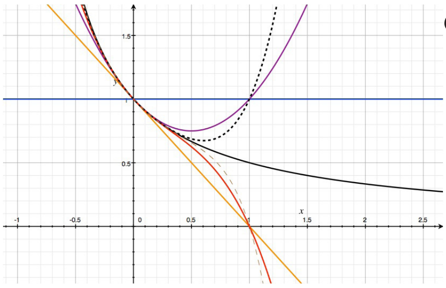
$$\Rightarrow \quad x = \frac{\beta}{\alpha}\left(1 - e^{-\alpha t}\right).$$

Now, as time increases, $e^{-\alpha t} \to 0$, and

$$x \to \frac{\beta}{\alpha}.$$

The height of a tree approaches $\frac{\beta}{\alpha}$ as time increases. The height will remain under β/α.

Exercise E.11.5

1. 

Quartic - short dots, quintic long dots.

2. $\frac{1}{1.1} \approx P_5(0.1) = 1 - 0.1 + 0.1^2 - 0.1^3 + 0.1^4 - 0.1^5 = .90909$ The error is $0.000000909...$

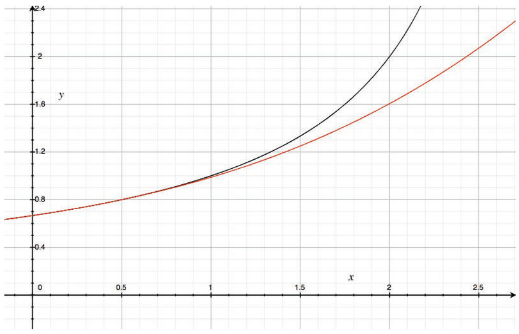
Exercise E.11.6

1. a $f(x) = \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$ b $f(x) = e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$

c $f(x) = \log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ...$

2 a $f(x) = e^{-x} = 1 - \frac{x^1}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - ...$ b $f(x) = \log_e(1-x) = x - \frac{x^2}{2} - \frac{x^3}{3} - ...$

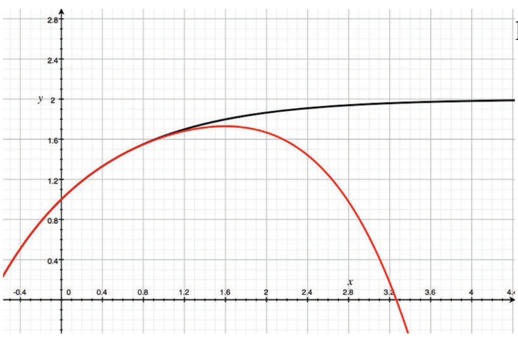
4. $\tan(x) \approx x + \frac{x^3}{3} + \frac{2x^5}{5}$

5. $\sum_{n=0}^{\infty} \frac{2x^n}{3^{n+1}}$ a $|x| < 3$ b 

approx. - red

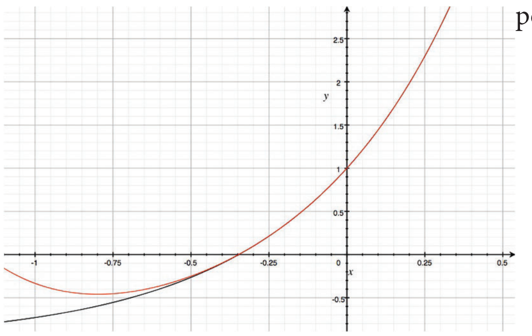
Exercise E.11.7

2. a $y(x) = 2 - e^{-x}$ b $p(x) = 1 + x - \frac{x^2}{2} + \frac{x^3}{3!} - \frac{x^4}{4!}$

c 

polynomial is red.

3. a $y(x) = -1 + 2e^{2x}$ b $p(x) = 1 + 4x + 4x^2 + \frac{8x^3}{3} + \frac{4x^4}{3}$

c 

polynomial red.

4. 0.080831 5. 0.22324

6. (a) $f(x) = \frac{1}{2-x} = \frac{1}{2} \cdot \frac{1}{1-x/2}$

$= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n, \quad \left|\frac{x}{2}\right| < 1$

$= \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \cdot x^n, \quad |x| < 2.$

(b) $f(x) = \frac{2}{1+4x} = 2 \cdot \frac{1}{1-(-4x)}$

$= 2 \sum_{n=0}^{\infty} (-4x)^n, \quad |4x| < 1$

$= \sum_{n=0}^{\infty} (-1)^n \, 2^{2n+1} \, x^n, \quad |x| < \frac{1}{4}.$

(c) $f(x) = \frac{3}{3+4x} = \frac{1}{1-\left(-\frac{4}{3}x\right)}$

$= \sum_{n=0}^{\infty} \left(-\frac{4}{3}\right)^n x^n, \quad \left|\frac{4}{3}x\right| < 1$

$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{4}{3}\right)^n x^n, \quad |x| < \frac{3}{4}.$

7. (a) $f(x) = \frac{1}{1-x^2} = \sum_{n=0}^{\infty} (x^2)^n, \quad |x^2| < 1$

$= \sum_{n=0}^{\infty} x^{2n}, \quad |x| < 1.$

(b) $f(x) = \dfrac{1}{4+x^2} = \dfrac{1}{4}\cdot\dfrac{1}{1+\frac{x^2}{4}} = \dfrac{1}{4}\cdot\dfrac{1}{1-\left(-\frac{x^2}{4}\right)}$

$$= \frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{x^2}{4}\right)^n, \quad \left|\frac{x^2}{4}\right|<1$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{4^n}, \quad |x^2|<4$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}x^{2n}, \quad |x|<2.$$

(c) $f(x) = \ln(5-x) = -\displaystyle\int\frac{1}{5-x}\,dx$

$$= -\frac{1}{5}\int\frac{1}{5-\frac{x}{5}}\,dx$$

$$= -\frac{1}{5}\int\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n dx$$

$$= -\frac{1}{5}\int\left(1+\frac{x}{5}+\frac{x^2}{5^2}+\frac{x^3}{5^3}+\dots\right)dx$$

$$= -\frac{1}{5}\left(x+\frac{x^2}{2.5}+\frac{x^3}{3.5^2}+\dots\right)+c$$

$$= -\frac{1}{5}\sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)5^n}+c, \quad \left|\frac{x}{5}\right|<1$$

Now, when $x=0$, we have $c=\ln(5)$.

Therefore, $f(x) = \ln(5-x)$

$$= \ln(5) - \sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)5^{n+1}}, \quad |x|<5.$$

8. (a) $f(x) = \frac{x}{1-x^2} = x \cdot \frac{1}{1-x^2}$

$= x \sum_{n=0}^{\infty} (x^2)^n, \quad |x^2| < 1$

$= \sum_{n=0}^{\infty} x^{2n+1}, \quad |x| < 1.$

(b) $f(x) = \frac{x^2}{x+2} = x^2 \cdot \frac{1}{2-(-x)} = \frac{x^2}{2} \cdot \frac{1}{1-\left(-\frac{x}{2}\right)}$

$= \frac{x^2}{2} \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n, \quad \left|\frac{x}{2}\right| < 1$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+2}, \quad |x| < 2.$

(c) $f(x) = \frac{x^2}{1+x^2} = x^2 \cdot \frac{1}{1-(-x^2)}$

$= x^2 \sum_{n=0}^{\infty} (-x^2)^n, \quad |x^2| < 1$

$= \sum_{n=0}^{\infty} (-1)^n x^{2(n+1)}, \quad |x| < 1.$

9. (a) i. $g(x) = \frac{1}{2+x} \Rightarrow g'(x) = -\frac{1}{(2+x)^2}.$

ii. $f(x) = \frac{1}{(2+x)^2} = -\frac{d}{dx}\left(\frac{1}{2+x}\right)$

$= -\frac{1}{2} \frac{d}{dx}\left(\frac{1}{1-\left(-\frac{x}{2}\right)}\right)$

$= -\frac{1}{2} \frac{d}{dx}\left(\sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n\right), \quad \left|\frac{x}{2}\right| < 1$

$$= -\frac{1}{2}\frac{d}{dx}\left(\frac{1}{2}-\frac{x}{2}+\frac{x^2}{2^2}-\frac{x^3}{2^3}+\frac{x^4}{2^4}-\dots\right)$$

$$= -\frac{1}{2}\left(-\frac{1}{2}+\frac{2x}{2^2}-\frac{3x^2}{2^3}+\frac{4x^3}{2^4}-\dots\right)$$

$$= \frac{1}{4}\left(1-\frac{2x}{2}+\frac{3x^2}{2^2}-\frac{4x^3}{2^3}+\dots\right)$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)x^n}{2^n}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)x^n}{2^{n+2}}, \quad |x|<2.$$

iii. $R = 2.$

(b) i. $h(x) = \frac{1}{(2+x)^3} = -\frac{1}{2}\frac{d}{dx}\left(\frac{1}{(2+x)^2}\right)$

$$= -\frac{1}{2}\frac{d}{dx}\left(\frac{1}{4}\left(1-\frac{2x}{2}+\frac{3x^2}{2^2}-\frac{4x^3}{2^3}+\frac{5x^4}{2^4}-\dots\right)\right), |x|<2$$

$$= -\frac{1}{8}\left(-1+\frac{2.3x}{2^2}-\frac{3.4x^2}{2^3}+\frac{4.5x^3}{2^4}-\dots\right)$$

$$= \frac{1}{2^3}\left(1-\frac{2.3x}{2^2}+\frac{3.4x^2}{2^3}-\frac{4.5x^3}{2^4}+\dots\right)$$

$$= \frac{1}{2^3}\sum_{n=0}^{\infty}(-1)^n\frac{(n+1)(n+2)x^n}{2^{n+1}}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{(n+1)(n+2)x^n}{2^{n+4}}, \quad |x|<2$$

ii. $q(x) = \frac{x^2}{(2+x)^3} = x^2 \cdot \frac{1}{(2+x)^3}$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{(n+1)(n+2)x^{n+2}}{2^{n+4}}, \quad |x|<2.$$

9. $$\int \frac{1}{1+x^7}\,dx = \int \frac{1}{1-(-x^7)}\,dx$$

$$= \int \sum_{n=0}^{\infty} (-x^7)^n \,dx$$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^{7n} \,dx$$

$$= \int \left(1 - x^7 + x^{14} - x^{21} + \ldots\right) dx$$

$$= x - \frac{x^8}{8} + \frac{x^{15}}{15} - \ldots + c$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+1}}{7n+1} + c.$$

10. (a) i. $$\ln(3+x) = \int \frac{1}{3+x}\,dx$$

$$= \frac{1}{3}\int \frac{1}{1-\left(-\frac{x}{3}\right)}\,dx$$

$$= \frac{1}{3}\int \sum_{n=0}^{\infty} \left(-\frac{x}{3}\right)^n dx$$

$$= \frac{1}{3}\int \left(1 - \frac{x}{3} + \frac{x^2}{3^2} - \frac{x^3}{3^3} + \ldots\right) dx$$

$$= \frac{1}{3}\left(x - \frac{x^2}{2.3} + \frac{x^3}{3.3^2} - \frac{x^4}{4.3^3} + \ldots\right) + c$$

$$= \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(n+1)3^n} + c, \quad \left|\frac{x}{3}\right| < 1.$$

Now, when $x = 0$, we have $c = \ln(3)$.

$$\Rightarrow \ln(3+x) = \ln(3) + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(n+1)3^{n+1}}, \quad |x| < 3$$

Let $k = n+1$: $$\ln(3+x) = \ln(3) + \sum_{k=1}^{\infty} (-1)^{k-1} \cdot \frac{x^k}{k.3^k}, \quad |x| < 3.$$

ii. $\underline{\underline{a = 3,\ b = 3}}$.

iii. Interval of convergence = $\underline{\underline{]-3, 3[}}$.

(b)

$$S_k(x) = \ln(a) + \sum_{n=1}^{k} (-1)^{n-1} \cdot \frac{1}{nb^n} x^n$$

$$= \ln(3) + \sum_{n=1}^{k} (-1)^{n-1} \cdot \frac{1}{n.3^n} x^n.$$

i. and ii. $f(x) = \ln(x+3)$.

$$S_2(x) = \ln(3) + \frac{x}{3} - \frac{x^2}{2.3^2}$$

$$S_5(x) = \ln(3) + \frac{x}{3} - \frac{x^2}{2.3^2} + \frac{x^3}{3.3^3} - \frac{x^4}{4.3^4} + \frac{x^5}{5.3^5}.$$

$$S_9(x) = \ln(3) + \frac{x}{3} - \frac{x^2}{2.3^2} + \frac{x^3}{3.3^3} - \frac{x^4}{4.3^4} + \frac{x^5}{5.3^5} - \frac{x^6}{6.3^6} + \frac{x^7}{7.3^7} - \frac{x^8}{8.3^8} + \frac{x^9}{9.3^9}.$$

Please turn over for graphs.

NOTE:

As n increases, the partial sums give a better approximation. We can see this on the graphs where, as n increases, the graph of the partial sum fits more closely to the graph of the function around $x = 0$.

10. (a)

$$\arctan(x) = \int \frac{1}{1+x^2}\,dx$$

$$= \int \frac{1}{1-(-x^2)}\,dx$$

$$= \int \sum_{n=0}^{\infty} (-x^2)^n \,dx, \qquad |x|<1$$

$$= \int \left(1 - x^2 + x^4 - x^6 + \dots\right) dx$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + c$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + c$$

and when $x=0$, $\arctan(x)=0 \Rightarrow c=0$.

So $$\frac{\arctan(x)}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n+1}, \quad |x|<1.$$

$$\int \frac{\arctan(x)}{x}\,dx = \int \left(1 - \frac{x^2}{3} + \frac{x^4}{5} - \dots\right) dx$$

$$= x - \frac{x^3}{3.3} + \frac{x^5}{5.5} - \dots + c$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)^2} + c, \quad |x|<1.$$

(b) $$\frac{\arctan(x^2)}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{2n+1}, \quad |x|<1$$

$$\int \frac{\arctan(x^2)}{x}\,dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{2n+1}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2(2n+1)^2} + c, \quad |x|<1.$$

(c) $x^2 \arctan(x^4) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{8n+6}}{2n+1}$

$$\int x^2 \arctan(x^4) = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{8n+6}}{2n+1} \, dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{8n+7}}{(2n+1)(8n+7)} + c, \quad |x|<1$$

12. $f(x) = \ln(1+x) = \int \frac{1}{1+x} \, dx$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^n \, dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + c$$

and $x = 0 \Rightarrow \ln(1+x) = 0 \Rightarrow c = 0$

We can also write $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$.

For an alternating series $|R_n| < u_{N+1}$,

So we need $u_{N+1} < 0.0001$ when $x = 0.5$.

$$\Rightarrow \frac{(0.5)^{N+1}}{N+1} < 0.0001.$$

Using a spreadsheet we find

$$u_8 = \frac{(0.5)^9}{9} = 0.0002\ldots > 0.0001.$$

$$u_9 = \frac{(0.5)^{10}}{10} = 0.000097\ldots < 0.0001.$$

So we need 9 terms in the series to estimate $\log_e(1.5)$ to within 0.0001.

13. $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$

(a) $f(x) = \sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} + \ldots$

$$= \underline{\underline{x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \ldots}}$$

(b) $\int_0^1 \sin(x^2)\,dx = \int_0^1 \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \ldots\right)dx$

$$= \left[\frac{x^3}{3} - \frac{x^7}{7.3!} + \frac{x^{11}}{11.5!} - \frac{x^{15}}{15.7!} + \ldots\right]_0^1$$

$$= \frac{1}{3} - \frac{1}{7.3!} + \frac{1}{11.5!} - \frac{1}{15.7!} + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)(2n+1)!}$$

Now $|R_{N+1}| < u_N = \frac{1}{(4N+3)(2N+1)!} \leq \frac{1}{19 \times 9!}$

Inspection shows $N = 4$ (from $4N+3 = 19$, $2N+1 = 9$)

$\therefore \int_0^1 \sin(x^2)\,dx \approx \frac{1}{3} - \frac{1}{7.3!} + \frac{1}{11.5!} - \frac{1}{15.7!}$

$$\approx \underline{\underline{0.3579}}$$

14. $f(x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)^3} = x + \frac{x^2}{2^3} + \frac{x^3}{3^3} + \frac{x^4}{4^3} + \cdots$

(a) $f'(x) = 1 + \frac{2x}{2^3} + \frac{3x^2}{3^3} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)^2}$

Now, $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+2)^2}\cdot\frac{(n+1)^2}{x^n}\right|$

$= \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^2|x| = |x| < 1.$

When $x = 1$: $\sum_{n=0}^{\infty}\frac{x^n}{(n+1)^2} = \sum_{n=0}^{\infty}\frac{1}{(n+1)^2}$: converges.

When $x = -1$: $\sum_{n=0}^{\infty}\frac{x^n}{(n+1)^2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}$: converges.

Interval of convergence is $[-1, 1]$.

(b) $f''(x) = \frac{1}{2^2} + \frac{2x}{3^2} + \frac{3x^2}{4^2} + \cdots = \sum_{0}^{\infty}\frac{(n+1)x^n}{(n+2)^2}$

Now, $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)x^{n+1}}{(n+3)^2}\cdot\frac{(n+2)^2}{(n+1)x^n}\right|$

$= \lim_{n\to\infty}\left(\frac{(n+2)^3}{(n+1)(n+3)}\right)|x| = |x| < 1$

When $x = 1$: $\sum_{n=0}^{\infty}\frac{(n+1)x^n}{(n+2)^2} = \sum_{n=0}^{\infty}\frac{n+1}{n^2+4n+4}$

$= \sum_{n=1}^{\infty}\frac{1+1/n}{n+4+4/n}$

$\sim \sum_{n=1}^{\infty}\frac{1}{n}$: diverges

When $x = -1$: $\sum_{n=0}^{\infty} \frac{(n+1)x^n}{(n+2)^2} = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{(n+1)^2}$

$= \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{n}{(n+1)^2}$

$= \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n+2+1/n}$: converges.

Interval of convergence is $[-1, 1[$.

(c) $f'''(x) = \frac{2}{3^2} + \frac{2.3x}{4^2} + \frac{3.4x}{5^2} + \ldots = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)x^n}{(n+3)^2}$

Now $\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{(n+2)(n+3)x^{n+1}}{(n+4)^2} \cdot \frac{(n+3)^2}{(n+1)(n+2)x^n} \right|$

$= \lim_{n\to\infty} \left(\frac{(n+3)^3}{(n+4)^2(n+1)} \right) |x|$

Expanding and dividing the numerator and denominator gives $\lim_{n\to\infty} (\ ) = 1$,

So we have $|x| < 1$.

When $x = 1$: $\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{(n+3)^2} = \sum_{n=0}^{\infty} \frac{n^2+3n+2}{n^2+6n+9}$

$= \sum_{n=0}^{\infty} \left(1 - \frac{3n+7}{n^2+6n+9}\right) = \infty$

$\Rightarrow$ diverges.

When $x = -1$: $\sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)}{(n+3)^2}$ diverges since $\lim_{n\to\infty} u_n \neq 0$.

Interval of convergence is $]-1, 1[$.

15. $f(x) = e^x$:

(a) Maclaurin's expansion is:

$$f(x) = f(0) + f'(0)x + f''(0).\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \dots$$

$$f(x) = e^x \Rightarrow f(0) = 1$$

$$f'(x) = f''(x) = \dots \Rightarrow f'(0) = f''(0) = \dots = 1$$

$$\Rightarrow f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

(b) Using the first 10 terms:

$$e^1 \approx 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{9!} \approx 2.71828152557$$

Actually, $e^1 \approx 2.71828182846$

Error $\approx 3.02886 \times 10^{-7}$

16. (a) $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$

$$\Rightarrow g(x) = \frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}, \quad |x| < 1.$$

(b)
$$h(x) = \frac{x}{(1-x^2)^2} = \frac{1}{2}\frac{d}{dx}\left(\frac{1}{1-x^2}\right)$$

$$= \frac{1}{2}\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^{2n}\right)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} 2nx^{2n-1}$$

$$= \sum_{n=0}^{\infty} nx^{2n-1}, \quad |x| < 1.$$

17. $f(x) = \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$, $|x|<1$

Now $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$ (see Qu.8 above)

$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

$\Rightarrow \ln(1-x) = -x - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \ldots$

$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots$

$\Rightarrow \ln(1+x) - \ln(1-x) = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \ldots$

$\ln\left(\frac{1+x}{1-x}\right) = \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}$, $|x|<1$.

18. $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$.

(a) $f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n^2} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}$

Now $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{x^n}{n+1} \cdot \frac{n}{x^{n-1}}\right|$

$= \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)|x| = |x| < 1$.

When $x = 1$: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

When $x = -1$: $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n^2}$ converges.

Interval of convergence is $[-1, 1]$.

(b) $f''(x) = \sum_{n=2}^{\infty} \frac{(n-1)x^{n-2}}{n}$ or $\sum_{n=1}^{\infty} \frac{nx^{n-1}}{n+1}$.

$$\lim_{n\to\infty}\left|\frac{(n+1)x^n}{n+2}\cdot\frac{n+1}{nx^{n-1}}\right| = \lim_{n\to\infty}\left(\frac{(n+1)^2}{n(n+2)}\right)|x| = |x| < 1.$$

When $x=1$: $\sum_{n=1}^{\infty}\frac{n}{n+1}$: diverges as $\lim_{n\to\infty} u_n \neq 0$.

When $x=-1$: $\sum_{n=1}^{\infty}(-1)^n\frac{n}{n+1}$:

The terms are decreasing but $\lim_{n\to\infty} u_n \neq 0$.

$\Rightarrow$ diverges.

Interval of convergence $= \underline{]-1,1[}$.

(c) $f'''(x) = \sum_{n=3}^{\infty}\frac{(n-1)(n-2)x^{n-3}}{n}$ or $\sum_{k=1}^{\infty}\frac{(k+2)(k+1)x^k}{k+3}$.

$$\lim_{k\to\infty}\left|\frac{(k+3)(k+2)x^{k+1}}{k+4}\cdot\frac{k+3}{(k+2)(k+1)x^k}\right|$$

$$= \lim_{k\to\infty}\left(\frac{(k+3)^2}{(k+4)(k+1)}\right)|x| \quad = |x| < 1.$$

When $x=1$: $\sum_{k=1}^{\infty}\frac{(k+2)(k+1)}{k+3} = \sum_{k=1}^{\infty}\frac{k^2+3k+2}{k+3}$

$= \sum_{k=1}^{\infty}\left(k+\frac{2}{k+3}\right)$: diverges.

When $x=-1$: $\sum_{k=1}^{\infty}(-1)^k\frac{(k+2)(k+1)}{k+3}$ and $\lim_{k\to\infty} u_k \neq 0$.

$\Rightarrow$ diverges.

Interval of convergence is $\underline{]-1,1[}$.

19. (a) $f(x)=\cos(x)$ $\quad f(0)=1$

$f'(x)=-\sin(x)$ $\quad f'(0)=0$

$f''(x)=-\cos(x)$ $\quad f''(0)=-1$

$f'''(x)=\sin(x)$ $\quad f'''(0)=0$

$f^{iv}(x)=\cos(x)$ $\quad f^{iv}(0)=1$ etc.

So $f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(x)}{3!}x^3+\ldots$

$$=1+0.x-\frac{1}{2!}x^2+0.x^3+\frac{1}{4!}x^4+\ldots$$

$$=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots=\underline{\underline{\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}}}.$$

(b) $\cos(\sqrt{x})=\sum_{n=0}^{\infty}\frac{(-1)^n(x^{\frac{1}{2}})^{2n}}{(2n)!}=\underline{\underline{\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{(2n)!}}}.$

(c) $\int_0^{0.1}\cos(\sqrt{x})\,dx=\int_0^{0.1}\left(1-\frac{x}{2!}+\frac{x^2}{4!}-\frac{x^3}{6!}+\ldots\right)dx$

$$=\left[x-\frac{x^2}{2.2!}+\frac{x^3}{3.4!}-\frac{x^4}{4.6!}+\ldots\right]_0^1$$

$$=1-\frac{1}{2.2!}+\frac{1}{3.4!}-\frac{1}{4.6!}+\ldots$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)(2n)!}.$$

Now $|R_N|<u_{N+1}=\frac{1}{(N+2)(2N)!}<5\times10^{-10}$.

Now, from the spreadsheet, $u_6\approx 2.98\times10^{-10}$

$u_7\approx 1.43\times10^{-12}$

$u_5\approx 4.6\times10^{-8}$.

And summing the first 6 terms $\approx \underline{\underline{0.7635}}$.

20. (a) $f(x) = e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|$$

$$= \lim_{n\to\infty}\left(\frac{1}{n+1}\right)|x| = 0.$$

Interval of convergence is $]-\infty, \infty[$.

$g(x) = \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2(n+1)}}{(2(n+1))!}\cdot\frac{(2n)!}{x^{2n}}\right|$$

$$= \lim_{n\to\infty}\left(\frac{1}{(2n+2)(2n+1)}\right)|x^2| = 0.$$

Interval of convergence is $]-\infty, \infty[$.

$h(x) = \ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{n+1}\cdot\frac{n}{(x-1)^n}\right|$$

$$= \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)|x-1|$$

$$= |x-1| < 1 \quad \Rightarrow \quad 0 < x < 2.$$

When $x = 0$: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(-1)^n = -\sum_{n=1}^{\infty}\frac{1}{n}$: diverges

When $x = 2$: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$: converges

Interval of convergence is $]0, 2]$.

(b) $u_x = e^{-x}\cos(x)$

$$= \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots - x + \frac{x^3}{2!} - \frac{x^5}{4!} + \frac{x^7}{6!} - \dots$$

$$+ \frac{x^2}{2!} - \frac{x^4}{2!2!} + \frac{x^6}{2!4!} - \dots - \frac{x^3}{3!} + \frac{x^5}{3!2!} - \frac{x^7}{3!4!} + \dots$$

$$+ \frac{x^4}{4!} - \frac{x^6}{4!2!} + \dots$$

$$= 1 - x + \frac{x^3}{3} - \frac{x^4}{6} + \frac{x^5}{24} - \frac{x^6}{720} + \dots$$

The interval of convergence is the intersection of the intervals for each series and e^{-x} and $\cos(x)$ are convergent on $\mathbb{R}$. So we have $]-\infty, \infty[$.

(c) $\left(\ln(x)\right)^2 = \left((x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \frac{(x-1)^6}{6} + \cdots\right) \times$

$\left((x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \frac{(x-1)^6}{6} + \cdots\right)$

$$= (x-1)^2 - \frac{(x-1)^3}{2} + \frac{(x-1)^4}{3} - \frac{(x-1)^5}{4} + \frac{(x-1)^6}{5} - \frac{(x-1)^7}{6} + \cdots$$
$$- \frac{(x-1)^3}{2} + \frac{(x-1)^4}{4} - \frac{(x-1)^5}{6} + \frac{(x-1)^6}{8} - \frac{(x-1)^7}{10} + \cdots$$
$$+ \frac{(x-1)^4}{3} - \frac{(x-1)^5}{6} + \frac{(x-1)^6}{9} - \frac{(x-1)^7}{12} + \cdots$$
$$- \frac{(x-1)^5}{4} + \frac{(x-1)^6}{8} - \frac{(x-1)^7}{12} + \cdots$$
$$+ \frac{(x-1)^6}{5} - \frac{(x-1)^7}{10} + \cdots$$
$$- \frac{(x-1)^7}{6} + \cdots$$
$$+ \cdots$$

$$= (x-1)^2 - (x-1)^3 + \frac{11}{12}(x-1)^4 - \frac{10}{12}(x-1)^5 + \frac{274}{360}(x-1)^6 - \frac{21}{30}(x-1)^7 + \cdots$$

$$\int_1^{1.25} (\ln(x))^2 \, dx$$

$$= \left[\frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{11}{60}(x-1)^5 - \frac{10}{60}(x-1)^6 + \frac{274}{6.360}(x-1)^7 - \frac{3}{30}(x-1)^8 + \cdots\right]_1^{1.25}$$

$$= \frac{1}{3}\left(\frac{1}{4}\right)^3 - \frac{1}{4}\left(\frac{1}{4}\right)^4 + \cdots \text{ etc}$$

Trial-and-error (via a spreadsheet) shows that the sixth term, $\left|\frac{1}{10}\left(\frac{1}{4}\right)^8\right| \approx 1.5\times10^{-6} < 5\times10^{-6}$

while fifth term, $\left|\frac{274}{6.360}\left(\frac{1}{4}\right)^7\right| \approx 7.7\times10^{-6}$

$> 5\times10^{-6}$.

Calculating the sum gives

$$\int_1^{1.25} (\ln(x))^2\, dx \approx \underline{\underline{0.0043786.}}$$

	Ex.3.4 Qu.8	Ex.3.4 Qu.9	19*9!	Ex.3.4 Qu.11	Ex.3.4 Qu.15	Ex.3.4 Qu.16	
0	0.5	0.333333333	1.45039E-07	1	1	0.25	0.005208333
1	-0.125	0.023809524		10	-0.25	0.25	-0.000976563
2	0.041666667	0.000757576		50	0.013888889	0.25	0.000179036
3	-0.015625	1.32275E-05		166.6666667	-0.000347222	0.25	-4.06901E-05
4	0.00625	1.45039E-07		416.6666667	4.96032E-06	0.25	7.74242E-06
5	-0.002604167			833.3333333	-4.59289E-08	0.25	-1.52588E-06
6	0.001116071			1388.888889	2.98239E-10		
7	-0.000488281			1984.126984	-1.43384E-12		
8	0.000217014			2480.15873			
9	-9.76563E-05			2755.731922			
10	4.43892E-05						
11	-2.03451E-05						
12	9.39002E-06						
13	-4.35965E-06						
14	2.03451E-06						
15	-9.53674E-07						
16	4.48788E-07						
17	-2.11928E-07						
18	1.00387E-07						
19	-4.76837E-08						
20	2.27065E-08						
21	-1.08372E-08						
22	5.18301E-09						
23	-2.48353E-09						
24	1.19209E-09						
25	-5.73122E-10						
26	2.75947E-10						
	S_9	**S_4**		**S_10**	**S_6**		**S_5**
	0.405532304	0.35791366		10086.57319	0.763546581		0.00437786
	ln(1.5)			**e^10**			
	0.405465108			22026.46579			
	Error			**Error**			
	-6.7196E-05			11939.8926			

Mathematics SL Errata

Mathematics SL Supplementary Material

Exercise A.3.1

1 Simplify the following.

e $\left(\frac{2x^3}{4y^2}\right)^2 \times \frac{12y^6}{8x^4}$ f $\frac{3^{n+2}+9}{3}$ g $\frac{4^{n+2}-16}{4}$ h $\frac{4^{n+2}-16}{2}$

i $\left(\frac{1}{2b}\right)^4 - \frac{b^2}{16}$

2 Simplify the following.

e $\frac{(xy)^6}{64x^6}$ f $\frac{27^{n+2}}{6^{n+2}}$

3 Simplify the following.

e $\frac{2^n \times 4^{2n+1}}{2^{1-n}}$ f $\frac{2^{2n+1} \times 4^{-n}}{(2^n)^3}$ g $\frac{x^{4n+1}}{(x^{n+1})^{(n-1)}}$ h $\frac{x^{4n^2+n}}{(x^{n+1})^{(n-1)}}$

i $\frac{(3^x)(3^{x+1})(3^2)}{(3^x)^2}$

5 Simplify the following, leaving your answer in positive power form.

e $\frac{(-2)^3 \times 2^{-3}}{(x^{-1})^2 \times x^2}$ f $\frac{(-a)^3 \times a^{-3}}{(b^{-1})^{-2}b^{-3}}$

6 Simplify the following.

e $\frac{(x-1)^{-3}}{(x+1)^{-1}(x^2-1)^2}$ f $\frac{y(x^{-1})^2 + x^{-1}}{x+y}$

7 Simplify the following.

a $5^{n+1} - 5^{n-1} - 2 \times 5^{n-2}$ b $a^{x-y} \times a^{y-z} \times a^{z-x}$ c $\left(\frac{a^{-\frac{1}{2}}b^3}{ab^{-1}}\right)^2 \times \frac{1}{ab}$

d $\left(\frac{a^{m+n}}{a^n}\right)^m \times \left(\frac{a^{n-m}}{a^n}\right)^{m-n}$ e $\frac{p^{-2}-q^{-2}}{p^{-1}-q^{-1}}$ f $\frac{1}{1+a^{\frac{1}{2}}} - \frac{1}{1-a^{\frac{1}{2}}}$

g $\frac{2^{n+4}-2(2^n)}{2(2^{n+3})}$ h $\sqrt{a\sqrt{a\sqrt{a}}}$

8 Simplify the following.

a $\frac{\sqrt{x} \times \sqrt[3]{x^2}}{\sqrt[4]{x}}$ b $\frac{b^{n+1} \times 8a^{2n-1}}{(2b)^2(ab)^{-n+1}}$ c $\frac{2^n - 6^n}{1-3^n}$

d $\frac{7^{m+1}-7^m}{7^n-7^{n+2}}$ e $\frac{5^{2n+1}+25^n}{5^{2n}+5^{1+n}}$

Exercise A.3.2

1. Solve the following equations.

g $\{x|\ 3^{2x-4} = 1\}$

h $\{x|\ 4^{2x+1} = 128\}$

i $\{x|\ 27^{x} = 3\}$

Exercise A.3.3

1 Use the definition of a logarithm to determine the following.

g $\log_4 1$ h $\log_{10} 1$ i $\log_{\frac{1}{2}} 2$ j $\log_{\frac{1}{3}} 9$

k $\log_3 \sqrt{3}$ l $\log_{10} 0.01$

3 Change the following logarithmic expression into its equivalent exponential form.

f $\log_2(ax - b) = y$

4 Solve for x in each of the following.

g $\log_x 16 = 2$ h $\log_x 81 = 2$

i $\log_x\left(\frac{1}{3}\right) = 3$ j $\log_2(x - 5) = 4$

k $\log_3 81 = x + 1$ l $\log_3(x - 4) = 2$

5 Solve for x in each of the following, giving your answer to 4 d.p.

g $\log_e(x + 2) = 4$ h $\log_e(x - 2) = 1$ i $\log_x e = -2$

Exercise A.3.4

1 Without using a calculator, evaluate the following.

e $\log_2 20 - \log_2 5$ f $\log_2 10 - \log_2 5$

2 Write down an expression for $\log a$ in terms of $\log b$ and $\log c$ for the following.

e $a = b^3c^4$ f $a = \frac{b^2}{\sqrt{c}}$

4 Express each of the following as an equation that does not involve a logarithm.

d $\log_2 x = y + 1$ e $\log_2 y = \frac{1}{2}\log_2 x$ f $3\log_2(x+1) = 2\log_2 y$

5 Solve the following equations.

d $\log_{10}(x+3) - \log_{10}x = \log_{10}x + \log_{10}2$

e $\log_{10}(x^2+1) - 2\log_{10}x = 1$

f $\log_2(3x^2+28) - \log_2(3x-2) = 1$

g $\log_{10}(x^2+1) = 1 + \log_{10}(x-2)$

h $\log_2(x+3) = 1 - \log_2(x-2)$

i $\log_6(x+5) + \log_6 x = 2$

j $\log_3(x-2) + \log_3(x-4) = 2$

k $\log_2 x - \log_2(x-1) = 3\log_2 4$

l $\log_{10}(x+2) - \log_{10}x = 2\log_{10}4$

6 Simplify the following

c $2\log_a x + 3\log_a(x+1)$ d $5\log_a x - \frac{1}{2}\log_a(2x-3) + 3\log_a(x+1)$

e $\log_{10}x^3 + \frac{1}{3}\log x^3y^6 - 5\log_{10}x$ f $2\log_2 x - 4\log_2\left(\frac{1}{y}\right) - 3\log_2 xy$

7 Solve the following

d $\log_3 x + \log_3(x-8) = 2$

e $\log_2 x + \log_2 x^3 = 4$

f $\log_3\sqrt{x} + 3\log_3 x = 7$

8 Solve for x.

c $\log_4 x^4 = (\log_4 x)^4$ d $\log_5 x^5 = (\log_5 x)^5$

e Investigate the solution to $\log_n x^n = (\log_n x)^n$.

9 Solve the following, giving an exact answer and an answer to 2 d.p.

e $3^{4x+1} = 10$ f $0.8^{x-1} = 0.4$ g $10^{-2x} = 2$

h $2.7^{0.3x} = 9$ i $0.2^{-2x} = 20$ j $\dfrac{2}{1+0.4^x} = 5$

k $\dfrac{2^x}{1-2^x} = 3$ l $\dfrac{3^x}{3^x+3} = \dfrac{1}{3}$

10 Solve for x.

c $\log_{10}(x^2-3x+6) = 1$ d $(\log_{10}x)^2 - 11\log_{10}x + 10 = 0$

e $\log_x(3x^2+10x) = 3$ f $\log_{x+2}(3x^2+4x-14) = 2$

11 Solve the following simultaneous equations.

c $xy = 2$
$2\log_2 x - \log_2 y = 2$

12 Express each of the following as an equation that does not involve a logarithm.

c $\ln x = y - 1$

13 Solve the following for x.

c $\log_e(x+1) + \log_e x = 0$ d $\log_e(x+1) - \log_e x = 0$

14 Solve the following for x.

c $-5 + e^{-x} = 2$ d $200e^{-2x} = 50$ e $\dfrac{2}{1-e^{-x}} = 3$

f $70e^{-\frac{1}{2}x} + 15 = 60$ g $\ln x = 3$ h $2\ln(3x) = 4$

i $\ln(x^2) = 9$ j $\ln x - \ln(x+2) = 3$ k $\ln\sqrt{x+4} = 1$ l $\ln(x^3) = 9$

15 Solve the following for x.

c $e^{2x} - 5e^{x} + 6 = 0$

d $e^{2x} - 2e^{x} + 1 = 0$

e $e^{2x} - 6e^{x} + 5 = 0$

f $e^{2x} - 9e^{x} - 10 = 0$

16 Solve each of the following.

a $4^{x-1} = 132$

b $5^{5x-1} = 3^{1-2x}$

c $3^{2x+1} - 7 \times 3^{x} + 4 = 0$

d $2^{2x+3} - 7 \times 2^{x+1} + 5 = 0$

e $3 \times 4^{2x+1} - 2 \times 4^{x+2} + 5 = 0$

f $3^{2x} - 3^{x+2} + 8 = 0$

g $2\log x + \log 4 = \log(9x - 2)$

h $2\log 2x - \log 4 = \log(2x - 1)$

i $\log_3 2x + \log_3 81 = 9$

j $\log_2 x + \log_x 2 = 2$

Exercise B.2.1

3 All of the following functions are mappings of $\mathbb{R} \mapsto \mathbb{R}$ unless otherwise stated.

a Determine the composite functions $(f \circ g)(x)$ and $(g \circ f)(x)$, if they exist.

b For the composite functions in part a that do exist, find their range.

viii $f(x) = x - 4, g(x) = |x|$

ix $f(x) = x^3 - 2, g(x) = |x + 2|$

x $f(x) = \sqrt{4 - x}, x \le 4, g(x) = x^2$

xi $f(x) = \dfrac{x}{x+1}, x \ne -1, g(x) = x^2$

xii $f(x) = x^2 + x + 1, g(x) = |x|$

xiii $f(x) = 2^x, g(x) = x^2$

xiv $f(x) = \dfrac{1}{x+1}, x \ne -1, g(x) = x - 1$

xv $f(x) = \dfrac{2}{\sqrt{x-1}}, x > 1, g(x) = x^2 + 1$

xvi $f(x) = 4^x, g(x) = \sqrt{x}$

13 Find $(h \circ f)(x)$, given that $h(x) = \begin{cases} x^2 + 4, & x \ge 1 \\ 4 - x, & x < 1 \end{cases}$ and $f: x \mapsto x - 1, x \in \mathbb{R}$.

Sketch the graph of $(h \circ f)(x)$ and use it to find its range.

14 a Given three functions, f, g and h, when would $h \circ g \circ f$ exist?

b If $f: x \mapsto x + 1, x \in \mathbb{R}$, $g: x \mapsto x^2, x \in \mathbb{R}$ and $h: x \mapsto 4x, x \in \mathbb{R}$, find $(h \circ g \circ f)(x)$.

15 Given the functions $f(x) = e^{2x-1}$ and $g(x) = \frac{1}{2}(\ln x + 1)$ find, where they exist:

a $(f \circ g)$ b $(g \circ f)$ c $(f \circ f)$

In each case find the range of the composite function.

16 Given that $h(x) = \log_{10}(4x - 1), x > \frac{1}{4}$ and $k(x) = 4x - 1, x \in\ ]-\infty, \infty[$, find, where they exist :

a $(h \circ k)$ b $(k \circ h)$.

17 Given the functions $f(x) = \sqrt{x^2 - 9}, x \in \mathbf{S}$ and $g(x) = |x| - 3, x \in \mathbf{T}$, find the largest positive subsets of $\mathbb{R}$ so that:

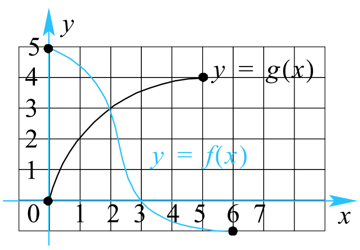
a $g \circ f$ exists b $f \circ g$ exists.

18 For each of the following functions:

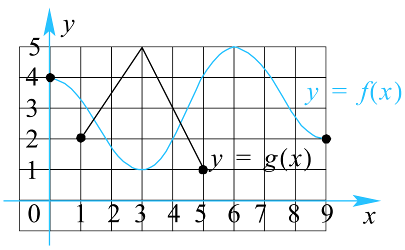
a determine if $f \circ g$ exists and sketch the graph of $f \circ g$ when it exists.

b determine if $g \circ f$ exists and sketch the graph of $g \circ f$ when it exists.

i



ii



19 Given the functions $f: \mathrm{S} \to \mathbb{R}$ where $f(x) = e^{x+1}$ and $g: \mathrm{S} \to \mathbb{R}$ where $g(x) = \ln 2x$ where $\mathrm{S} =]0, \infty[$.

a State the domain and range of both f and g.

b Giving reasons, show that $g\circ f$ exists but $f\circ g$ does not exist.

c Fully define $g\circ f$, sketch its graph and state its range.

20 The functions f and g are given by $f(x) = \begin{cases} \sqrt{x-1} & \text{if } x \geq 1 \\ x-1 & \text{if } 0 < x < 1 \end{cases}$ and $g(x) = x^2 + 1$.

a Show that $f\circ g$ is defined. b Find $(f\circ g)(x)$ and determine its range.

21 Let $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$ where $f(x) = \begin{cases} \dfrac{1}{x^2}, & 0 < x \leq 1 \\ \dfrac{1}{\sqrt{x}}, & x > 1 \end{cases}$.

a Sketch the graph of f.

b Define the composition $f\circ f$, justifying its existence.

c Sketch the graph of $f\circ f$, giving its range.

22 Consider the functions $f :]1,\infty[\rightarrow \mathbb{R}$ where $f(x) = \sqrt{x}$ and $g : \mathbb{R}\backslash\{0\} \rightarrow \mathbb{R}$ where $g(x) = x^2$.

a Sketch the graphs of f and g on the same set of axes.

b Prove that $g\circ f$ exists and find its rule.

c Prove that $f\circ g$ cannot exist.

d If a new function $g^* : S \rightarrow \mathbb{R}$ where $g^*(x) = g(x)$ is now defined, find the largest positive subset of $\mathbb{R}$ so that $f\circ g^*$ does exist. Find $f\circ g^*$, sketch its graph and determine its range.

23 Given that $f(x) = \dfrac{ax-b}{cx-a}$, show that $f\circ f$ exists and find its rule.

24 a Sketch the graphs of $f(x) = \dfrac{1}{a}x^2$ and $g(x) = \sqrt{2a^2 - x^2}$, where $a > 0$.

b Show that $f\circ g$ exists, find its rule and state its domain.

c Let S be the largest subset of $\mathbb{R}$ so that $g\circ f$ exists.

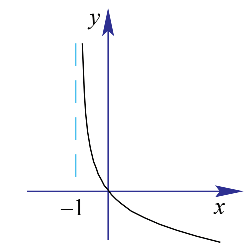
i Find S.

ii Fully define $g\circ f$, sketch its graph and find its range.

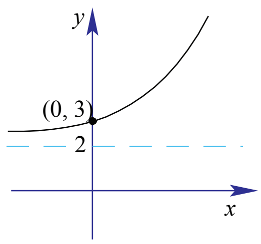
Exercise B.2.2

5 Sketch the inverse of the following functions.

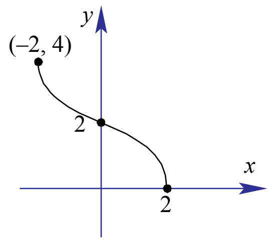
e



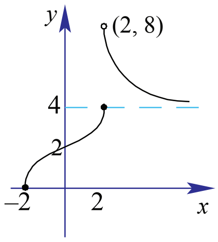
f



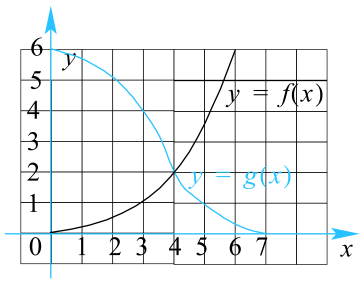
g



h



18 Consider the functions f and g:



a Does $g \circ f$ exist? Justify your answer.

b Does $(g \circ f)^{-1}$ exist? Justify your answer.

If it does exist, sketch the graph of $(g \circ f)^{-1}$.

19 a On the same set of axes, sketch the graph of $f(x) = -x^3$ and its inverse, $f^{-1}(x)$.

b The function g is given by $g(x) = \begin{cases} 2x+1, & x < -1 \\ -x^3, & -1 \le x \le 1 \\ 2x-1, & x > 1 \end{cases}$.

i Sketch the graph of g.

ii Fully define its inverse, g^{-1}, stating why it exists.

iii Sketch the graph of g^{-1}.

iv Find $\{x : g(x) = g^{-1}(x)\}$.

20 Consider the functions $t(x) = e^x$ and $m(x) = \sqrt{x}$.

a Find, where they exist, the composite functions: **i** $(t \circ m)(x)$ **ii.** $(m \circ t)(x)$

b With justification, find and sketch the graphs of: **i** $(t \circ m)^{-1}(x)$ **ii** $(m \circ t)^{-1}(x)$

c Find: **i** $t^{-1} \circ m^{-1}(x)$ **ii** $m^{-1} \circ t^{-1}(x)$

d What conclusion(s) can you make from your results of parts **b** and **c**?

e Will your results of part **d** work for any two functions f and g? Explain.

21 **a** Find $\{x : x^3 + x - 2 = 0\}$.

b If $f(x) = \frac{1}{\sqrt{x}} - 2$, sketch the graph of $y = f(x)$ and find $\{x : f(x) = f^{-1}(x)\}$.

22 Consider the functions $f(x) = |x|, x \in \mathbf{A}$ and $g(x) = e^x - 2, x \in \mathbf{B}$.

a Sketch the graphs of: **i** f if $\mathbf{A} = \mathbb{R}$ **ii** g if $\mathbf{B} = \mathbb{R}$.

b With **A** and **B** as given in part **a**, give reasons why $(f \circ g)^{-1}$ will not exist.

c **i** Find the largest set **B** which includes positive values, so that $(f \circ g)^{-1}$ exists.

ii Fully define $(f \circ g)^{-1}$.

iii On the same set of axes, sketch the graphs of $(f \circ g)(x)$ and $(f \circ g)^{-1}(x)$.

Exercise B.4.5

6. Consider the function $f(x) = \frac{2-x}{2+x}$.

a i Find the coordinates of the intercepts with the axes.

ii Determine the equations of the asymptotes of f.

iii Hence, sketch the graph of f.

iv Determine the domain and range of f.

b Find f^{-1} , the inverse function of f.

b Deduce the graph of $(f(x))^2$.

7. Express $\frac{8x-5}{x-3}$ in the form $A + \frac{B}{x-3}$, where A and B are integers.

a Hence, state the equations of the vertical and horizontal asymptotes of the function $f(x) = \frac{8x-5}{x-3}$.

b Sketch the graph of $f(x) = \frac{8x-5}{x-3}$ and use it to determine its range.

8. On different sets of axes, sketch the graphs of $f(x) = 2 + \frac{1}{x}$ and $g(x) = \frac{1}{f(x)}$, stating their domains and ranges.

9. Sketch the graphs of the following functions, clearly labelling all asymptotes.

a $f(x) = 2x + \frac{1}{x}, x \neq 0$ b $g(x) = \frac{1}{2}x + \frac{1}{x^2}, x \neq 0$

c $g(x) = -x + \frac{1}{x}, x \neq 0$ d $f(x) = x - \frac{1}{x}, x \neq 0$

10. Sketch the graphs of the following functions, clearly labelling all asymptotes.

a $h(x) = x^2 + \frac{2}{x}, x \neq 0$ b $f(x) = x^2 + \frac{1}{x^2}, x \neq 0$

c $g(x) = x - \frac{1}{x^2}, x \neq 0$ d $f(x) = x^3 + \frac{3}{x}, x \neq 0$

11. Sketch the graphs of the following functions, clearly labelling all asymptotes.

a $f(x) = x + 3 + \frac{2}{x}, x \neq 0$ b $f(x) = -x + \frac{1}{x} + 2, x \neq 0$

c $g(x) = 2x + \frac{1}{x^2} - 2, x \neq 0$ d $f(x) = \frac{x^2 + 2x - 2}{x}, x \neq 0$

12. a For the function $f(x) = 3 + \frac{1}{1-x} - x$:

i determine all axial intercepts and the coordinates of its stationary points.

ii write down the equation of all the asymptotes.

b Sketch the graph of $y = f(x)$ clearly labelling all the information from part a.

13. Sketch the graphs of: a $f(x) = \frac{x^2 - x - 1}{x - 2}, x \neq 2$. b $g(x) = \frac{(x+2)^2(x-1)}{x^2}, x \neq 0$.

14. Sketch the graphs of the following functions.

a $f(x) = \dfrac{2x-3}{x^2-3x+2}$ b $y = \dfrac{x^2+2x}{x^2+4}$ c $y = \dfrac{x^4+1}{x^2+1}$.

15. Sketch the graph of $f(x) = \dfrac{x+1}{\sqrt{x-1}}$, clearly identifying all asymptotes and turning points.

Exercise E.6.2

9. The acceleration, in m/s² of a body in a medium is given by $\frac{dv}{dt} = \frac{3}{t+1}, t \geq 0$. The particle has an initial speed of 6 m/s, find the speed (to 2 d.p) after 10 seconds.

10. The rate of change of the water level in an empty container, t seconds after it started to be filled from a tap is given by the relation:

$$\frac{dh}{dt} = 0.2\sqrt[3]{t+8}, t \geq 0$$
,

where h cm is the water level. Find the water level after 6 seconds.

11. The gradient function of the curve $y = f(x)$ is given by $e^{0.5x} - \cos(2x)$. Find the equation of the function, given that it passes through the origin.

12. a Given that $\frac{d}{dx}(e^{ax}(p\sin bx + q\cos bx)) = e^{ax}\sin bx$, express p and q in terms of a and b.

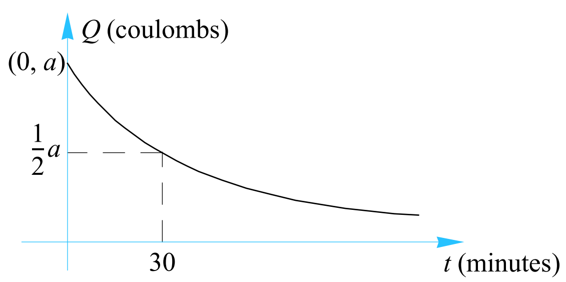
b Hence find $\int e^{2x}\sin 3x dx$.

13 The rate of change of the charge, Q, in coulombs, retained by a capacitor t minutes after charging, is given by $\frac{dQ}{dt} = -ake^{-kt}$.

Using the graph shown, determine the charge remaining after

a one hour

b 80 minutes



14. a Show that $\frac{d}{dx}(x\ln(x+k)) = \frac{x}{x+k} + \ln(x+k)$, where k is a real number.

b For a particular type of commercial fish, it is thought that a length–weight relationship exists such that their rate of change of weight, w kg, with respect to their length, x m, is modelled by the equation:

$$\frac{dw}{dx} = 0.2\ln(x+2).$$

Given that a fish in this group averages a weight of 650 gm when it is 20 cm long, find the weight of a fish measuring 30 cm.

15. The rate of flow of water, $\frac{dV}{dt}$ litres/hour, pumped into a hot water system over a 24-hour period from 6:00 am, is modelled by the relation:

$$\frac{dV}{dt} = 12 + \frac{3}{2}\cos\frac{\pi}{3}t,\ t \geq 0.$$

a Sketch the graph of $\frac{dV}{dt}$ against t.

b For what percentage of the time will the rate of flow exceed 11 litres/hour.

c How much water has been pumped into the hot water system by 8:00 a.m.?

16. The rates of change of the population size of two types of insect pests over a 4-day cycle, where t is measured in days, has been modelled by the equations:

$$\frac{dA}{dt} = 2\pi\cos\pi t,\ t \geq 0 \text{ and } \frac{dB}{dt} = \frac{3}{4}e^{0.25t},\ t \geq 0$$

where A and B represent the number of each type of pest in thousands.

Initially there were 5000 insects of type A and 3000 insects of type B.

a On the same set of axes sketch the graphs, $A(t)$ and $B(t)$ for $0 \leq t \leq 4$.

b What is the maximum number of insects of type A that will occur?

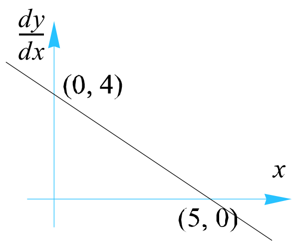
c When will there first be equal numbers of insects of both types?

d For how long will the number of type B insects exceed the number of type A insects during the four days?

Exercise E.6.4

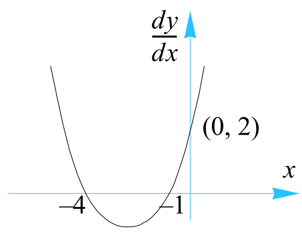
11. Sketch the graph of $y = f(x)$ for each of the following:

a



Where the curve passes through the point (5, 10).

b



Where the curve passes through the point (0, 0).

12. Find $f(x)$ given that $f''(x) = 12x + 4$ and that the gradient at the point (1, 6) is 12.

13. Find $f(x)$ given that $f'(x) = ax^2 + b$, where the gradient at the point (1, 2) is 4, and that the curve passes through the point (3, 4).

14. The rate at which a balloon is expanding is given by

$$\frac{dV}{dt} = kt^{4.5}, t \geq 0,$$

where t is the time in minutes since the balloon started to be inflated and V cm^3 is its volume. Initially the balloon (which may be assumed to be spherical) has a radius of 5 cm. If the balloon has a volume of 800 cm^3 after 2 minutes, find its volume after 5 minutes.

15. The area, A cm^2, of a healing wound caused by a fall on a particular surface decreases at a rate given by the equation:

$$A'(t) = -\frac{35}{\sqrt{t}}$$

where t is the time in days. Find the initial area of such a wound if after one day the area measures 40 cm^2.

Exercise E.6.5

6. Evaluate the following definite integrals (giving exact values).

g $\int_0^1 \frac{2}{(x+1)^3}dx$ h $\int_2^4 \left(\sqrt{x} - \frac{2}{\sqrt{x}}\right)^2 dx$ i $\int_3^4 \frac{2x+1}{2x^2-3x-2}dx$

8. Evaluate the following definite integrals (giving exact values).

e $\int_0^{\frac{\pi}{4}} (x - \sec^2 x)dx$ f $\int_0^{\frac{\pi}{2}} 2\cos\left(4x + \frac{\pi}{2}\right)dx$ g $\int_{-\pi}^{\pi} \left(\sin\left(\frac{x}{2}\right) + 2\cos(x)\right)dx$

h $\int_0^{\frac{\pi}{12}} \sec^2\left(\frac{\pi}{4} - 2x\right)dx$ i $\int_0^{\pi} \cos(2x+\pi)dx$

13. a Find $\frac{d}{dx}(xe^{0.1x})$. Hence, find $\int xe^{0.1x}dx$.

b Following an advertising initiative by the Traffic Authorities, preliminary results predict that the number of alcohol-related traffic accidents has been decreasing at a rate of $-12 - te^{0.1t}$ accidents per month, where t is the time in months since the advertising campaign started.

i How many accidents were there over the first six months of the campaign?

ii In the year prior to the advertising campaign there were 878 alcohol-related traffic accidents. Find an expression for the total number of accidents since the start of the previous year, t months after the campaign started.

14. The rate of cable television subscribers in a city t years from 1995 has been modelled by the equation $\frac{2000}{\sqrt{(1+0.4t)^3}}$.

a How many subscribers were there between 1998 and 2002?

b If there were initially 40 000 subscribers, find the number of subscribers by 2010.

15.

a Find $\frac{d}{dt}\left(\frac{800}{1+24e^{-0.02t}}\right)$.

b The rate at which the number of fruit flies appear when placed in an environment with limited food supply in an experiment was found to be approximated by the exponential model:

$$\frac{384e^{-0.02t}}{(1+24e^{-0.02t})^2}, t \geq 0$$

where t is the number of days since the experiment started. What was the increase in the number of flies after 200 days?

Exercise E.6.6

20. Find the area of the region bounded by the curves with equations $y = \sqrt{x}$, $y = 6 - x$ and the x-axis.

21. a Sketch the graph of the function $f(x) = |e^x - 1|$.

b Find the area of the region enclosed by the curve $y = f(x)$,

i the x-axis and the lines $x = -1$ and $x = 1$.

ii the y-axis and the line $y = e - 1$.

iii and the line $y = 1$. Discuss your findings for this case.

22. a On the same set of axes, sketch the graphs of $f(x) = \sin\left(\frac{1}{2}x\right)$ and $g(x) = \sin 2x$ over the interval $0 \le x \le \pi$.

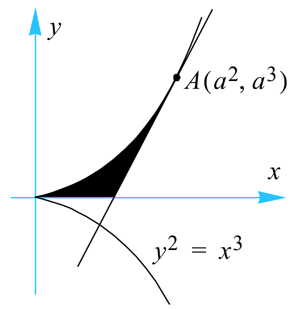
b Find the area of the region between by the curves $y = f(x)$ and $y = g(x)$ over the interval $0 \le x \le \pi$, giving your answer correct to two decimal places.

23. Consider the curve with equation $y^2 = x^3$ as shown in the diagram.

A tangent meets the curve at the point $A(a^2, a^3)$.

a Find the equation of the tangent at A.

b Find the area of the shaded region enclosed by the curve, the line $y = 0$ and the tangent.



24. a On a set of axes, sketch the graph of the curve $y = e^{x-1}$ and find the area of the region enclosed by the curve, the x-axis and the lines $x = 0$ and $x = 1$.

b Hence evaluate $\int_{e^{-1}}^{1} (\ln x + 1)\,dx$.

c Find the area of the region enclosed by the curves $y = e^{x-1}$ and $y = \ln x + 1$ over the $e^{-1} \le x \le 1$.

Exercise E.11.1

8. Use Euler's method, with a step size of 0.5, to calculate an approximate value for $y(4)$ for the solution of $\frac{dy}{dx} = (x + y - 1)^2$ given that the curve passes through the point $(0, 2)$.

9. Use Euler's method, with a step size of 0.1, to calculate an approximate value for $y(1)$ for the solution of $\frac{dy}{dx} = x^2 + y^2$ given that the curve passes through the point $(0, 1)$.

10. Given the initial-value problem, $\frac{dy}{dx} = y - y^2$, $y(0) = 0.5$, find an approximate value of $y(1)$ using Euler's method with step size $h = 0.2$.

11. Given the initial-value problem, $\frac{dy}{dx} = y + y^2$, $y(0) = 0.5$, find an approximate value of $y(1)$ using Euler's method with step size $h = 0.2$.

12. Use Euler's method, with a step size of 0.1, to calculate an approximate value for $y(2.6)$ given that $\frac{dy}{dx} = 3x - 2y + 1$ and that the curve passes through the point $(2, 5)$.

13. (a) Sketch the slope field along with the solution curve to the initial-value problem, $\frac{dy}{dx} = x + y^2$, $y(0) = 1$.
    (b) Using Euler's method with step size $h = 0.2$, calculate an approximate value of $y(1)$.

Exercise B.4.6

7. On the same set of axes, sketch the graphs of $f(x) = 5 \times 5^{-x}$ and $g(x) = 5^x - 4$.

   Find: i $\{(x, y) : f(x) = g(x)\}$ ii $\{x : f(x) > g(x)\}$.

8. Find the range of the following functions.

   a $f:]0,\infty[\mapsto \mathbb{R}$, where $f(x) = e^{-(x+1)} + 2$.

   b $g(x) = -2 \times e^x + 1, x \in]-\infty,0]$.

   c $x \mapsto xe^{-x} + 1, x \in [-1,1]$

9. a Sketch the graph of $f(x) = |2^x - 1|$, clearly labelling all intercepts with the axes and the equation of the asymptote.

   b Solve for x, where $|2^x - 1| = 3$.

10. Sketch the graphs of the following functions:

    a $f(x) = |1 - 2^x|$ b $g(x) = |4^x - 2|$ c $h(x) = 1 - |2^x|$

11. Sketch the graphs of the following functions.

    a $f(x) = 1 - 2^{-|x|}$ b $g(x) = -4 + 2^{|x|}$ c $h(x) = |3^{-|x|} - 3|$

12. Sketch the graphs of the following functions and find their range.

    a $f(x) = \begin{cases} 2^x, & x < 1 \\ 3, & x \geq 1 \end{cases}$ b $f(x) = \begin{cases} 3 - e^x, & x > 0 \\ x + 3, & x \leq 0 \end{cases}$

    c $f(x) = \begin{cases} \dfrac{2}{x+1}, & x \geq 1 \\ 3 - 2^{2-x}, & x < 1 \end{cases}$ d $g(x) = \begin{cases} 4 - 3^{-|x|}, & -1 < x < 1 \\ 4 - \frac{1}{3}|x|, & 1 \leq |x| \leq 12 \end{cases}$

13. Sketch the graphs of the following, and hence state the range in each case.

    a $f: \mathbb{R} \mapsto \mathbb{R}\ y = 2^x + \left(\frac{1}{2}\right)^x$ b $f: \mathbb{R} \mapsto \mathbb{R}\ y = 3^x + \left(\frac{1}{3}\right)^x$

    c $f: \mathbb{R} \mapsto \mathbb{R}\ y = 2^x - \left(\frac{1}{2}\right)^x$ d $f: \mathbb{R} \mapsto \mathbb{R}\ y = \left|2^x - \left(\frac{1}{2}\right)^x\right|$

14. Sketch the graph of the functions.

    a $g(x) = 2^{(x-a)}, a > 0$ b $h(x) = 2^x - a, 0 < a < 1$

    c $f(x) = 2 \times a^x - 2a, a > 1$ d $f(x) = 2 \times a^x - 2a, 0 < a < 1$

    e $g(x) = a - a^x, a > 1$ f $h(x) = -a + a^{-x}, a > 1$

15. a On the same set of axes, sketch $f(x) = 2 \times a^x$ and $g(x) = 4 \times a^{-x}$ where a > 1.

    Hence, sketch the graph of the function $h(x) = a^x + 2a^{-x}$, where a > 1.

    b On the same set of axes, sketch $f(x) = x - a$ and $g(x) = a^{x+1}$, where $a > 1$.

    Hence, deduce the graph of $h(x) = (x - a) \times a^{x+1}$, where $a > 1$.

16. Sketch the graph of the following functions and determine their range.

a $f(x) = a^{-x^2}, a > 1$

b $f(x) = a^{-x^2}, 0 < a < 1$

c $g(x) = (a-1)^{-x}, a > 1$

d $h(x) = 2 - a^{-|x|}, a > 1$

e $f(x) = \frac{2}{a^x - 1}, a > 1$

f $g(x) = |a^{x^2} - a|, a > 1$

Exercise B.4.7

6. Given the function $y = f(x)$, sketch the graphs of:

   a $y = |f(x)|$ b $y = f(|x|)$

   c $f(x) = \log_{10}(-x)$ d $f(x) = \ln\left(\frac{1}{x} - e\right)$ p

   e $f(x) = 2 - \ln(ex - 1)$ f $f(x) = \log_2(x^2 - 2x)$

7. a On the same set of axes, sketch the graphs of $f(x) = \ln x - 1$ and $g(x) = \ln(x - e)$.

   b Find $\{x : \ln x > \ln(x - e) + 1\}$.

8. Sketch the graphs of the following functions and find their ranges.

   a $f(x) = \begin{cases} \log_{10}x, & x \geq 1 \\ 1 - x, & x < 1 \end{cases}$ b $f(x) = \begin{cases} \log_2(x^2 - 1), & |x| \geq 1 \\ 1 - x^2, & |x| < 1 \end{cases}$

   c $f(x) = \begin{cases} 2 - \ln x, & x \geq e \\ \frac{x^3}{e^3}, & x < e \end{cases}$ d $g(x) = \begin{cases} 1 + \sqrt{x - 1}, & x > 1 \\ |\log_2 x| + 1, & 0 < x \leq 1 \\ 1 & x \leq 0 \end{cases}$

9. Sketch the graphs of the following functions.

   a $f(x) = \log_{\frac{1}{2}}x$ b $f(x) = \log_{\frac{1}{2}}(x - 2)$ c $f(x) = \log_{\frac{1}{3}}x + 1$

10. Sketch the graph of the following functions, clearly stating domains and labelling asymptotes.

    a $f(x) = 2\log_a(x - a), a > 1$ b $f(x) = -\ln(ax - e), a > e$

    c $g(x) = |\log_{10}(10 - ax)|, 1 < a < 10$ d $g(x) = \ln|x - ae|, a > 1$

    e $g(x) = |\ln|x - ae||, a > 1$ f $h(x) = \log_a\left(1 - \frac{x}{a}\right), 0 < a < 1$

11. Sketch the graph of $f(x) = \frac{1}{a}\log_a(ax - 1), 0 < a < 1$ clearly labelling its asymptote, and intercept(s) with the axes.

    Hence, find $\left\{x : f(x) > \frac{1}{a}\right\}$.

12. Sketch the graph of: a $f(x) = \frac{\ln x}{x}, x > 0$ b $g(x) = \frac{x}{\ln x}, x > 0$

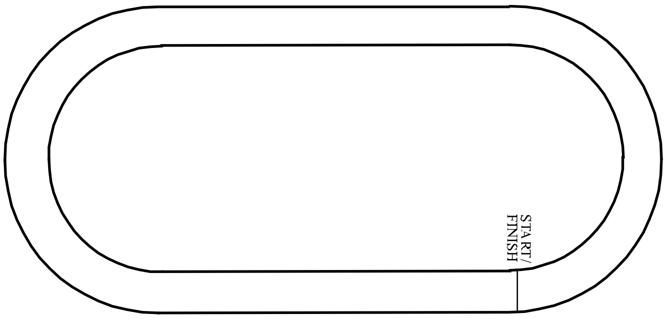
    Given that $f(x) \leq e^{-1}$ for all real $x > 0$, state the range of $g(x)$.

Exercise C.4.1

1. Find the areas and perimeters of the following sectors.

	Radius	Angle
h	8.6 cm	$\frac{7\pi}{6}$
i	6.2 cm	$\frac{4\pi}{3}$
j	76 m	$\frac{11\pi}{6}$
k	12 cm	30°
l	14 m	60°
m	2.8 cm	120°
n	24.8 cm	270°
o	1.2 cm	15°

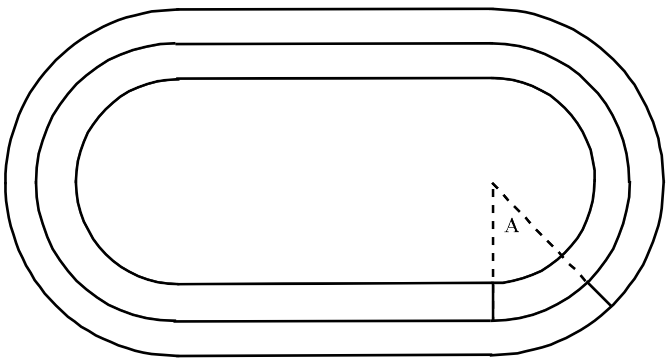
8.. The diagram shows a running track. The perimeter of the inside line is 400 metres and the length of each straight section is 100 metres.



a Find the radius of each of the semicircular parts of the inner track.

b If the width of the lane shown is 1 metre, find the perimeter of the outer boundary of the lane.

A second lane is added on the outside of the track. The starting positions of runners who have to run (anticlockwise) in the two lanes are shown.



c Find the value of angle A° (to the nearest degree) if both runners are to run 400 metres.

9. Find the angle subtended by at the centre of radius length 12 cm which forms a sector of area 80 sq. cm.

10. Find the angle subtended by an arc of a circle of radius length 10 cm which forms a sector of area 75 sq. cm.

11. A chord of length 32 cm is drawn in a circle of radius 20 cm.

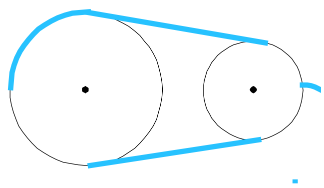
    a Find the angle it subtends at the centre.

    b Find: **i** the minor arc length **ii** the major arc length.

    c Find the area of the minor sector.

12. Two circles of radii 6 cm and 8 cm have their centres 10 cm apart. Find the area common to both circles.

13.. Two pulleys of radii 16 cm and 20 cm have their centres 40 cm apart. Find the length of the piece of string that will be required to pass tightly round the circles if the string does not cross over.



14. Two pulleys of radii 7 cm and 11 cm have their centres 24 cm apart. Find the length of the piece of string that will be required to pass tightly round the circles if:

    a the string cannot cross over.

    b the string crosses over itself.

15. A sector of a circle has a radius of 15 cm and an angle of 216°. The sector is folded in such a way that it forms a cone, so that the two straight edges of the sector do not overlap.

    a Find the base radius of the cone.

    b Find the vertical height of the cone.

    c Find the semi–vertical angle of the cone.

16. A taut belt passes over two discs of radii 4 cm and 12 cm as shown in the diagram.

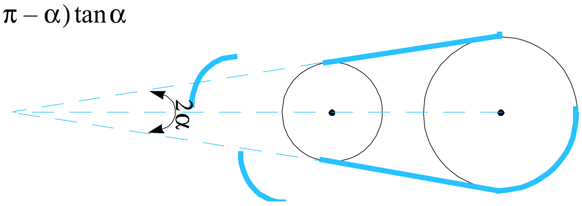
    a If the total length of the belt is 88 cm, show that $1 = (5.5 - \pi - \alpha)\tan\alpha$

    b On the same set of axes, sketch the graphs of:

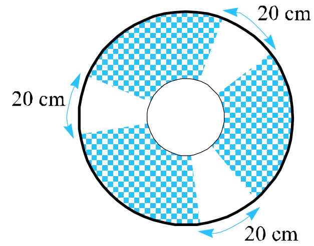
    i $y = \dfrac{1}{\tan\alpha}$

    ii $y = 5.5 - \pi - \alpha$.

    c Hence find $\{\alpha : 1 = (5.5 - \pi - \alpha)\tan\alpha\}$, giving your answer to two d.p.



17. The diagram shows a disc of radius 40 cm with parts of it painted. The smaller circle (having the same centre as the disc) has a radius of 10 cm. What area of the disc has not been painted in blue?

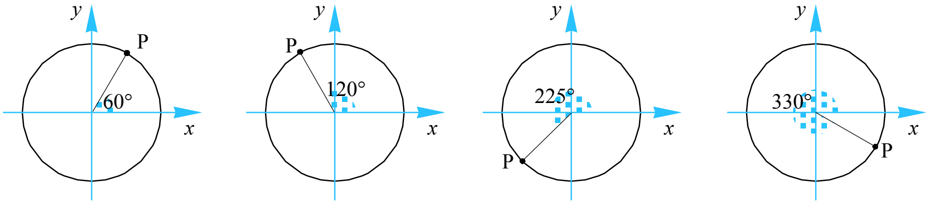


Exercise C.5.1

1

7. Find the coordinates of the point P on the following unit circles.

a **b** **c** **d**



8. Find the exact value of:

a $\sin\frac{11\pi}{6}\cos\frac{5\pi}{6} - \sin\frac{5\pi}{6}\cos\frac{11\pi}{6}$ **b** $2\sin\frac{\pi}{6}\cos\frac{\pi}{6}$

c $\dfrac{\tan\frac{\pi}{3} - \tan\frac{\pi}{6}}{1 + \tan\frac{\pi}{3}\tan\frac{\pi}{6}}$ **d** $\cos\frac{\pi}{4}\cos\frac{\pi}{3} + \sin\frac{\pi}{4}\sin\frac{\pi}{3}$

9. Show that the following relationships are true.

a $\sin 2\theta = 2\sin\theta\cos\theta$, where $\theta = \frac{\pi}{3}$ **b** $\cos 2\theta = 2\cos^2\theta - 1$, where $\theta = \frac{\pi}{6}$

c $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$, where $\theta = \frac{2\pi}{3}$ **d** $\sin(\theta - \phi) = \sin\theta\cos\phi - \sin\phi\cos\theta$, where $\theta = \frac{2\pi}{3}$ and $\phi = \frac{\pi}{3}$

10. Given that $\sin\theta = \frac{2}{3}$ and $0 < \theta < \frac{\pi}{2}$, find:

a $\sin(\pi + \theta)$ **b** $\sin(2\pi - \theta)$ **c** $\cos\left(\frac{\pi}{2} + \theta\right)$

11. Given that $\cos\theta = \frac{2}{5}$ and $0 < \theta < \frac{\pi}{2}$, find:

a $\cos(\pi - \theta)$ **b** $\sec\theta$ **c** $\sin\left(\frac{\pi}{2} - \theta\right)$

12. Given that $\tan\theta = k$ and $0 < \theta < \frac{\pi}{2}$, find:

a $\tan(\pi + \theta)$ **b** $\tan\left(\frac{\pi}{2} + \theta\right)$ **c** $\tan(-\theta)$

13. Given that $\sin\theta = \frac{2}{3}$ and $0 < \theta < \frac{\pi}{2}$, find:

a $\cos\theta$ **b** $\sec\theta$ **c** $\cos(\pi+\theta)$

14. Given that $\cos\theta = -\frac{4}{5}$ and $\pi < \theta < \frac{3\pi}{2}$, find:

a $\sin\theta$ **b** $\tan\theta$ **c** $\cos(\pi+\theta)$

15. Given that $\tan\theta = -\frac{4}{3}$ and $\frac{\pi}{2} < \theta < \pi$, find:

a $\sin\theta$ **b** $\tan\left(\frac{\pi}{2}+\theta\right)$ **c** $\sec\theta$

16. Given that $\cos\theta = k$ and $\frac{3\pi}{2} < \theta < 2\pi$, find:

a $\cos(\pi-\theta)$ **b** $\sin\theta$ **c** $\cot\theta$

17. Given that $\sin\theta = -k$ and $\pi < \theta < \frac{3\pi}{2}$,

find:

a $\cos\theta$ **b** $\tan\theta$ **c** $\operatorname{cosec}\left(\frac{\pi}{2}+\theta\right)$

18. Simplify the following.

a $\dfrac{\sin(\pi-\theta)\cos\left(\frac{\pi}{2}+\theta\right)}{\sin(\pi+\theta)}$ **b** $\dfrac{\sin\left(\frac{\pi}{2}+\theta\right)\cos\left(\frac{\pi}{2}-\theta\right)}{\sin^2\theta}$ **c** $\dfrac{\sin\left(\frac{\pi}{2}-\theta\right)}{\cos\theta}$

d $\tan(\pi+\theta)\cot\theta$ **e** $\cos(2\pi-\theta)\operatorname{cosec}\theta$ **f** $\dfrac{\sec\theta}{\operatorname{cosec}\theta}$

19. If $0 \le \theta \le 2\pi$, find all values of x such that:

a $\sin x = \frac{\sqrt{3}}{2}$ **b** $\cos x = \frac{1}{2}$ **c** $\tan x = \sqrt{3}$

d $\cos x = -\frac{\sqrt{3}}{2}$ **e** $\tan x = -\frac{1}{\sqrt{3}}$ **f** $\sin x = -\frac{1}{2}$

Exercise C.5.2

6. Prove $\sin^2x(1+n\cot^2x)+\cos^2x(1+n\tan^2x) = \sin^2x(n+\cot^2x)+\cos^2x(n+\tan^2x)$.

7. If $k\sec\phi = m\tan\phi$, prove that $\sec\phi\tan\phi = \dfrac{mk}{m^2-k^2}$.

8. If $x = k\sec^2\phi + m\tan^2\phi$ and $y = l\sec^2\phi + n\tan^2\phi$, prove that $\dfrac{x-k}{k+m} = \dfrac{y-l}{l+n}$.

9. Given that $\tan\theta = \dfrac{2a}{a^2-1}$, $0<\theta<\dfrac{\pi}{2}$, find: **a** $\sin\theta$ **b** $\cos\theta$

10. **a** If $\sin x + \cos x = 1$, find the values of: **i** $\sin^3x+\cos^3x$ **ii** $\sin^4x+\cos^4x$

b Hence, deduce the value of $\sin^k x + \cos^k x$, where k is a positive integer.

11. If $\tan\phi = -\dfrac{1}{\sqrt{x^2-1}}$, $\dfrac{\pi}{2}<\phi<\pi$, find, in terms of x,

a $\sin\phi+\cos\phi$ b $\sin\phi-\cos\phi$ c $\sin^4\phi-\cos^4\phi$

12. Find: **a** the maximum value of **b** the minimum value of

i $\cos^2\theta+5$ **ii** $\dfrac{5}{3\sin^2\theta+2}$ **iii** $2\cos^2\theta+\sin\theta-1$

13. **a** Given that $b\sin\phi = 1$ and $b\cos\phi = \sqrt{3}$, find b.

b Hence, find all values of ϕ that satisfy the relationship described in part **a**.

14. Find: **a** the maximum value of **b** the minimum value of

i $5^{3\sin\theta-1}$ **ii** $3^{1-2\cos\theta}$

15. Given that $\sin\theta\cos\theta = k$, find: **a** $(\sin\theta+\cos\theta)^2$, $\sin\theta+\cos\theta>0$.

b $\sin^3\theta+\cos^3\theta$, $\sin\theta+\cos\theta>0$ b

16. **a** Given that $\sin\phi = \frac{1-a}{1+a}, 0 < \phi < \frac{\pi}{2}$, find $\tan\phi$.

b Given that $\sin\phi = 1-a, \frac{\pi}{2} < \phi < \pi$, find : **i** $2 - \cos\phi$ **ii** $\cot\phi$

17. Find:

a the value(s) of $\cos x$, where $\cot x = 4(\operatorname{cosec} x - \tan x)$, $0 < x < \pi$.

b the values of $\sin x$, where $3\cos x = 2 + \frac{1}{\cos x}, 0 \le x \le 2\pi$.

18. Given that $\sin 2x = 2\sin x\cos x$, find all values of x, such that $2\sin 2x = \tan x, 0 \le x \le \pi$.

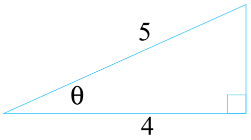
Extra Examples

Example 3.3.8

If $\sin\theta = \frac{3}{5}$ and $\cos\phi = -\frac{12}{13}$, where $0 \le \theta \le \frac{\pi}{2}$ and $\pi \le \phi \le \frac{3\pi}{2}$, find:

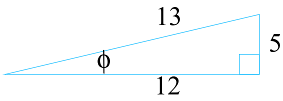
a $\sin(\theta + \phi)$ b $\cos(\theta + \phi)$ c $\tan(\theta - \phi)$

We start by drawing two right-angled triangles satisfying the given conditions:



$\cos\theta = \frac{4}{5}$

$\tan\theta = \frac{3}{4}$



$\sin\phi = \frac{5}{13}$

$\tan\phi = \frac{5}{12}$

$\sin(\theta + \phi) = \sin\theta\cos\phi + \sin\phi\cos\theta$

However, we cannot simply substitute the above ratios into this expression as we now need to consider the sign of the ratios.

As $0 \le \theta \le \frac{\pi}{2}$ then $\cos\theta = \frac{4}{5}$ and as $\pi \le \phi \le \frac{3\pi}{2}$ then $\sin\phi = -\frac{5}{13}$.

Therefore, $\sin(\theta + \phi) = \frac{3}{5} \times -\frac{12}{13} + -\frac{5}{13} \times \frac{4}{5} = -\frac{56}{65}$

$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

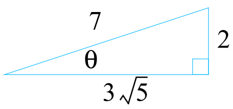
As $0 \le \theta \le \frac{\pi}{2}$ then $\cos\theta = \frac{4}{5}$ and as $\pi \le \phi \le \frac{3\pi}{2}$ then $\sin\phi = -\frac{5}{13}$.

Therefore, $\cos(\theta + \phi) = \frac{4}{5} \times -\frac{12}{13} - \frac{3}{5} \times -\frac{5}{13} = -\frac{33}{65}$

Example 3.3.9

If $\sin\theta = \frac{2}{7}$, where $\frac{\pi}{2} \le \theta \le \pi$, find: a $\sin 2\theta$ b $\cos 2\theta$ c $\tan 2\theta$

We start by drawing the relevant right-angled triangle:



a $\sin 2\theta = 2\sin\theta\cos\theta = 2 \times \frac{2}{7} \times -\frac{3\sqrt{5}}{7}$

$= -\frac{12\sqrt{5}}{49}$

b $\cos 2\theta = 1 - 2\sin^2\theta = 1 - 2 \times \left(\frac{2}{7}\right)^2 = \frac{41}{49}$

c $\tan 2\theta = \dfrac{\sin 2\theta}{\cos 2\theta} = \dfrac{-\frac{12\sqrt{5}}{49}}{\frac{41}{49}} = -\dfrac{12\sqrt{5}}{41}$

Example 3.3.10

Prove that: a $\sin 2\alpha \tan\alpha + \cos 2\alpha = 1$ b $2\cot 2\beta = \cot\beta - \tan\beta$

a L.H.S $= \sin 2\alpha \tan\alpha + \cos 2\alpha = 2\sin\alpha\cos\alpha \times \dfrac{\sin\alpha}{\cos\alpha} + (1 - 2\sin^2\alpha)$

$= 2\sin^2\alpha + 1 - 2\sin^2\alpha$ = R.H.S

$= 1$

b R.H.S $= \cot\beta - \tan\beta = \dfrac{\cos\beta}{\sin\beta} - \dfrac{\sin\beta}{\cos\beta}$

$= \dfrac{\cos^2\beta - \sin^2\beta}{\sin\beta\cos\beta}$

$= \dfrac{\cos 2\beta}{\frac{1}{2}\sin 2\beta}$

$= 2\dfrac{\cos 2\beta}{\sin 2\beta}$

$= 2\cot 2\beta$

= L.H.S

Notice that, when proving identities, when all else fails, then express everything in terms of sine and cosine. This will always lead to the desired result – even though sometimes the working seems like it will only grow and grow – eventually, it does simplify. Be persistent.

To prove a given identity, any one of the following approaches can be used:

1. Start with the L.H.S and then show that L.H.S = R.H.S

2. Start with the R.H.S and then show that R.H.S = L.H.S

3. Show that L.H.S = p, show that R.H.S = p $\Rightarrow$ L.H.S = R.H.S

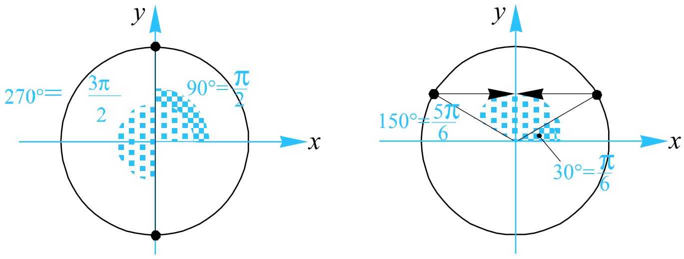
4. Start with L.H.S = R.H.S $\Rightarrow$ L.H.S – R.H.S = 0 .

When using approaches 1 and 2, choose whichever side has more to work with.

Example 3.3.11

Find all values of x, such that $\sin 2x = \cos x$, where $0 \le x \le 2\pi$.

$\sin 2x = \cos x \Leftrightarrow 2\sin x\cos x = \cos x$



$$\Leftrightarrow 2\sin x\cos x - \cos x = 0$$

$$\Leftrightarrow \cos x(2\sin x - 1) = 0$$

$$\Leftrightarrow \cos x = 0 \text{ or } \sin x = \frac{1}{2}$$

Now, $\cos x = 0, 0 \le x \le 2\pi \Leftrightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$

and

$\sin x = \frac{1}{2}, 0 \le x \le 2\pi \Leftrightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$

Example 3.3.12

Simplify $\sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$.

Express $\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$, where R and α are real numbers.

Hence find the maximum value of $\cos\theta - \sin\theta$.

$$\sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right) = \sqrt{2}\left[\sin\theta\cos\frac{\pi}{4} + \cos\theta\sin\frac{\pi}{4}\right] = \sqrt{2}\left[\sin\theta \times \frac{1}{\sqrt{2}} + \cos\theta \times \frac{1}{\sqrt{2}}\right]$$

$$= \sin\theta + \cos\theta$$

In this instance, as the statement needs to be true for all values of θ, we will determine the values of R and α by setting $R\cos(\theta + \alpha) \equiv \cos\theta - \sin\theta$.

Now, $R\cos(\theta + \alpha) = R[\cos\theta\cos\alpha - \sin\theta\sin\alpha] = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

Therefore, we have that $R\cos\theta\cos\alpha - R\sin\theta\sin\alpha \equiv \cos\theta - \sin\theta$

$\Rightarrow R\cos\theta\cos\alpha = \cos\theta \Leftrightarrow R\cos\alpha = 1$ – (1)

$\Rightarrow R\sin\theta\sin\alpha = \sin\theta \Leftrightarrow R\sin\alpha = 1$ – (2)

Dividing (2) by (1) we have $\frac{R\sin\alpha}{R\cos\alpha} = \frac{1}{1} \Leftrightarrow \tan\alpha = 1 \therefore \alpha = \frac{\pi}{4}$

Substituting into (1) we have $R\cos\frac{\pi}{4} = 1 \Leftrightarrow R \times \frac{1}{\sqrt{2}} = 1 \therefore R = \sqrt{2}$.

Therefore, $\cos\theta - \sin\theta \equiv \sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right)$

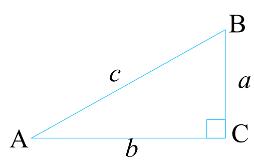
Then, as the maximum value of the cosine is 1, the maximum of $\sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right)$ is $\sqrt{2}$.

Exercise C.5.3

10. Prove that: **a** $\frac{1 + \sin x + \cos x}{1 + \sin x - \cos x} = \cot\frac{1}{2}x$ **b** $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$

c $\sin^4\phi = \frac{3}{8} + \frac{1}{8}\cos 4\phi - \frac{1}{2}\cos 2\phi$ **d** $\sin x = \frac{2\tan\frac{1}{2}x}{1 + \tan^2\frac{1}{2}x}$

11. For the right-angled triangle shown, prove that:



a $\sin 2\alpha = \frac{2ab}{c^2}$ **b** $\cos 2\alpha = \frac{b^2 - a^2}{c^2}$

c $\sin\frac{1}{2}\alpha = \sqrt{\frac{c - b}{2c}}$ **d** $\cos\frac{1}{2}\alpha = \sqrt{\frac{c + b}{2c}}$

12. Find the exact value $\tan\frac{\pi}{8}$.

13. Given that $\alpha + \beta + \gamma = \pi$, prove that $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4\sin\alpha\sin\beta\sin\gamma$.

14. Solve the following for $0 \le x \le 2\pi$.

a $\sin x = \sin 2x$ **b** $\sin x = \cos 2x$ **c** $\tan 2x = 4\tan x$

15. **a** Given that $a\sin\theta + b\cos\theta \equiv R\sin(\theta + \alpha)$, express R and α in terms of a and b.

b Find the maximum value of $5 + 4\sin\theta + 3\cos\theta$.

16. **a** Given that $a\cos\theta + b\sin\theta \equiv R\cos(\theta - \alpha)$, express R and α in terms of a and b.

b Find the minimum value of $2 + 12\cos\theta + 5\sin\theta$.

17. Prove that $\tan\left(\frac{\pi}{4} + \frac{1}{2}x\right) = \sec x + \tan x$.

18. Show that if $t = \tan\frac{\pi}{12}$ then $t^2 + 2\sqrt{3}t = 1$. Hence find the exact value of $\tan\frac{\pi}{12}$.

Exercise C.6.1

7. Sketch graphs of the following functions for x-values in the interval $[-2\pi, 2\pi]$.

a $y = \sin(2x)$

b $y = -\cos\left(\frac{x}{2}\right)$

c $y = 3\tan\left(x - \frac{\pi}{4}\right)$

d $y = 2\sin\left(x - \frac{\pi}{2}\right)$

e $y = 1 - 2\sin(2x)$

f $y = -2\cos\left(\frac{x - \pi}{2}\right)$

g $y = 3\tan\left[\frac{1}{2}\left(x + \frac{\pi}{4}\right)\right] - 3$

h $y = 3\cos\left(x + \frac{\pi}{4}\right)$

i $y = 2\sin\left[\frac{1}{3}\left(x + \frac{2\pi}{3}\right)\right] - 1$

j $y = 3\tan(2x + \pi)$

k $y = 4\sin\left(\frac{x + \frac{\pi}{2}}{2}\right)$

l $y = 2 - \sin\left(\frac{2(x - \pi)}{3}\right)$

m $y = 2\cos(\pi x)$

n $y = 2\sin[\pi(x + 1)]$

8.

a i Sketch one cycle of the graph of the function $f(x) = \sin x$.

ii For what values of x is the function $y = \frac{1}{f(x)}$ not defined?

iii Hence, sketch one cycle of the graph of the function $g(x) = \operatorname{cosec} x$.

bi Sketch one cycle of the graph of the function $f(x) = \cos x$.

ii For what values of x is the function $y = \frac{1}{f(x)}$ not defined?

iii Hence, sketch one cycle of the graph of the function $g(x) = \sec x$.

ci Sketch one cycle of the graph of the function $f(x) = \tan x$.

ii For what values of x is the function $y = \frac{1}{f(x)}$ not defined?

iii Hence, sketch one cycle of graph of the function $g(x) = \cot x$.

Exercise C.7.1

6. Solve the following equations for the intervals indicated, giving exact answers:

e $\cos^2 x = 2\cos x, -\pi \le x \le \pi$

f $\sec 2x = \sqrt{2}, 0 \le x \le 2\pi$

g $2\sin^2 x - 3\cos x = 2, 0 \le x \le 2\pi$

h $\sin 2x = 3\cos x, 0 \le x \le 2\pi$

7. Find:

a $3\tan^2 x + \tan x = 2, 0 \le x \le 2\pi$.

b $\tan^3 x + \tan^2 x = 3\tan x + 3, 0 \le x \le 2\pi$.

8. If $0 \le x \le 2\pi$, find:

a $\sin^2 2x - \frac{1}{4} = 0$ b $\tan^2\left(\frac{x}{2}\right) - 3 = 0$ c $\cos^2(\pi x) = 1$

9. If $0 \le x \le 2\pi$, find:

a $\sec^2 x + 2\sec x = 8$ b $\sec^2 x = 2\tan x + 4$

c $\cot^2 x - \sqrt{3}\cot x = 0$ d $6\operatorname{cosec}^2 x = 8 + \cot x$

10. a Express $\sqrt{3}\sin x + \cos x$ in the form $R\sin(x + \alpha)$.

b Solve $\sqrt{3}\sin x + \cos x = 1, 0 \le x \le 2\pi$.

11. a Express $\sin x - \sqrt{3}\cos x$ in the form $R\sin(x + \alpha)$.

b Solve $\sin x - \sqrt{3}\cos x = -1, 0 \le x \le 2\pi$.

12. Find x if $2\sin\left(x + \frac{\pi}{3}\right) + 2\sin\left(x - \frac{\pi}{3}\right) = \sqrt{3}, 0 \le x \le 2\pi$.

13. a Sketch the graph of $f(x) = \sin x, 0 \le x \le 4\pi$.

b Hence, find: i $\left\{x \mid \sin x > \frac{1}{2}\right\} \cap \{x \mid 0 < x < 4\pi\}$.

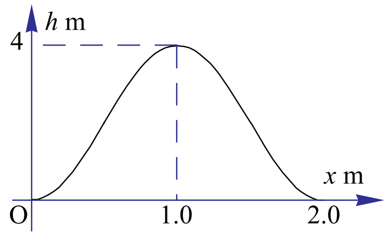
ii $\{x \mid \sqrt{3}\sin x < -1\} \cap \{x \mid 0 < x < 4\pi\}$.

Exercise C.7.2

14. A hill has its cross-section modelled by the function,

$$h : [0, 2] \mapsto \mathbb{R},\ h(x) = a + b\cos(kx),$$

where $h(x)$ measures the height of the hill relative to the horizontal distance x m from O.



a Determine the values of

i k

ii b

iii a

b How far, horizontally from O, would an ant climbing this hill from O be, when it first reaches a height of 1 metre?

c How much further, horizontally, will the ant have travelled when it reaches the same height of 1 metre once over the hill and on its way down?

15. A nursery has been infested by two insect pests: the Fruitfly and the Greatfly. These insects appear at about the same time that a particular plant starts to flower. The number of Fruitfly (in thousands), t weeks after flowering has started is modelled by the function

$$F(t) = 6 + 2\sin(\pi t),\ 0 \le t \le 4$$

Whereas the number of Greatfly (in thousands), t weeks after flowering has started is modelled by the function

$$G(t) = 0.25t^2 + 4,\ 0 \le t \le 4$$

a Copy and complete the following table of values, giving your answers correct to the nearest hundred.

t	0	0.5	1	1.5	2	2.5	3	3.5	4
$F(t)$									
$G(t)$									

b On the same set of axes **draw** the graphs of:

i $F(t) = 6 + 2\sin(\pi t),\ 0 \le t \le 4$. ii $G(t) = 0.25t^2 + 4,\ 0 \le t \le 4$.

c On how many occasions will there be equal numbers of each insect?

d For what percentage of the time will there be more Greatflies than Fruitflies?

16. The depth, $d(t)$ metres, of water at the entrance to a harbour at t hours after midnight on a particular day is given by

$$d(t) = 12 + 3\sin\left(\frac{\pi}{6}t\right),\ 0 \le t \le 24$$

a Sketch the graph of $d(t)$ for $0 \le t \le 24$.

b For what values of t will: i $d(t) = 10.5,\ 0 \le t \le 24$ ii $d(t) \ge 10.5,\ 0 \le t \le 24$.

Boats requiring a minimum depth of b metres are only permitted to enter the harbour when the depth of water at the entrance of the harbour is at least b metres for a continuous period of one hour.

c Find the largest value of b, correct to two decimal place, which satisfies this condition.

Exercise E.2.3

3. Differentiate the following.

g $\frac{4}{x^2} \times \sin x$ h $xe^x \sin x$ i $xe^x \log_e x$

4. Differentiate the following.

g $\frac{e^x - 1}{x + 1}$ h $\frac{\sin x + \cos x}{\sin x - \cos x}$ i $\frac{x^2}{x + \log_e x}$

5. Differentiate the following.

f $\cos(-4x) - e^{-3x}$ g $\log_e(4x + 1) - x$ h $\log_e(e^{-x}) + x$

i $\sin\left(\frac{x}{2}\right) + \cos(2x)$ j $\sin(7x - 2)$ k $\sqrt{x} - \log_e(9x)$

l $\log_e(5x) - \cos(6x)$

6. Differentiate the following.

i $\cos(\sin\theta)$ j $4\sec\theta$ k $\operatorname{cosec}(5x)$ l $3\cot(2x)$

7. Differentiate the following.

k $e^{-\cos(2\theta)}$ l $e^{2\log_e(x)}$ m $\frac{2}{e^{-x} + 1}$

n $(e^x - e^{-x})^3$ o $\sqrt{e^{2x + 4}}$ p $e^{-x^2 + 9x - 2}$

8. Differentiate the following.

i $\log_e\left(\frac{1}{\sqrt{x + 2}}\right)$ j $\log_e(\cos^2 x + 1)$ k $\log_e(x\sin x)$ l $\log_e\left(\frac{x}{\cos x}\right)$

9. Differentiate the following.

i $\frac{\cos(2x)}{e^{1 - x}}$ j $x^2\log_e(\sin 4x)$ k $e^{-\sqrt{x}}\sin\sqrt{x}$ l $\cos(2x\sin x)$

m $\frac{e^{5x + 2}}{1 - 4x}$ n $\frac{\log_e(\sin\theta)}{\cos\theta}$ o $\frac{x}{\sqrt{x + 1}}$ p $x\sqrt{x^2 + 2}$

q $(x^3 + x)^3\sqrt{x + 1}$ r $(x^3 - 1)\sqrt{x^3 + 1}$ s $\frac{1}{x}\log_e(x^2 + 1)$

t $\log_e\left(\frac{x^2}{x^2 + 2x}\right)$ u $\frac{\sqrt{x - 1}}{x}$ v $e^{-x}\sqrt{x^2 + 9}$

w $(8-x^3)\sqrt{2-x}$ x $x^n\ln(x^n-1)$

15. Find: c $\frac{d}{dx}(\cos x°)$

17. a Given that $f(x) = 1-x^3$ and $g(x) = \log_e x$, find: i $(f \circ g)'(x)$ ii $(g \circ f)'(x)$

b Given that $f(x) = \sin(x^2)$ and $g(x) = e^{-x}$, find: i $(f \circ g)'(x)$ ii $(g \circ f)'(x)$

21. Differentiate the following.

e $y = \cot\left(\frac{\pi}{4}-x\right)$ f $y = \sec(2x-\pi)$

22. Differentiate the following.

g $x^4\operatorname{cosec}(4x)$ h $\tan 2x \cot x$ i $\sqrt{\sec x + \cos x}$

23. Differentiate the following.

a $e^{\sec x}$ b $\sec(e^x)$ c $e^x\sec x$

d $\cot(\ln x)$ e $\ln(\cot 5x)$ f $\cot x \ln x$

g $\operatorname{cosec}(\sin x)$ h $\sin(\operatorname{cosec} x)$ i $\sin x \operatorname{cosec} x$

Exercise E.2.4

1. Find the second derivative of the following functions.

m $f(x) = x^3 \sin x$ n $y = x \ln x$ o $f(x) = \dfrac{x^2 - 1}{2x + 3}$ p $y = x^3 e^{2x}$

q $f(x) = \dfrac{\cos(4x)}{e^x}$ r $y = \sin(x^2)$ s $f(x) = \dfrac{x}{1 - 4x^3}$ t $y = \dfrac{x^2 - 4}{x - 3}$

7. Find the *n*th derivative of:

a e^{ax} b $y = \dfrac{1}{2x + 1}$ c $\sin(ax + b)$

8. a Find $f''(2)$ if $f(x) = x^2 - \sqrt{x}$. b Find $f''(1)$ if $f(x) = x^2 \mathrm{Tan}^{-1}(x)$.

9. Find the rate of change of the gradient of the function $g(x) = \dfrac{x^2 - 1}{x^2 + 1}$ where $x = 1$.

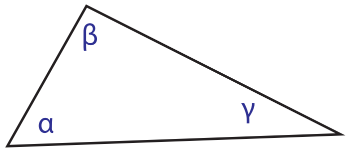
10. Find the values of x where the rate of change of the gradient of the curve $y = x \sin x$ for $0 \le x \le 2\pi$ is positive.

Important Geometry Theorems

(Not a complete list of MYP Geometry Theorems!)

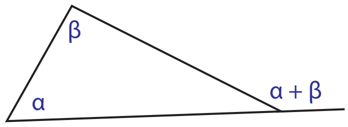
1. ANGLES OF A TRIANGLE

The sum of the measures of the angles of a triangle is 180˚.



2. THE EXTERIOR ANGLE OF A TRIANGLE

The exterior angle of a triangle is equal in size to the sum of the interior opposite angles.

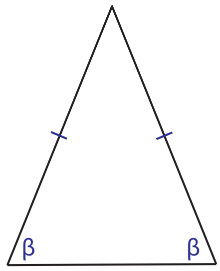


3. THE SUM OF THE ANGLES OF A POLYGON

The sum of the measure of the angles of an n-sided polygon is $(n-2)\times180^\circ$

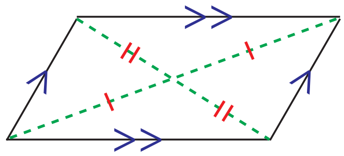
4. THE BASE ANGLES OF AN ISOSCELES TRIANGLE

The base angle of an isosceles triangle are equal.



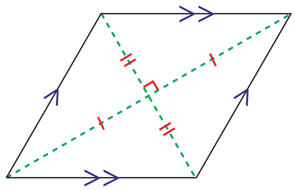
5. THE DIAGONALS OF A PARALLELOGRAM

The diagonals of a parallelogram bisect each other.



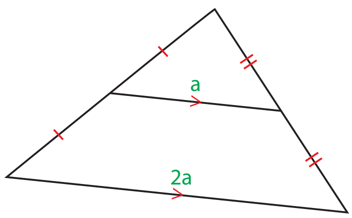
6. THE DIAGONALS OF A RHOMBUS and its CONVERSE

The diagonals of a rhombus bisect each other at right angles.



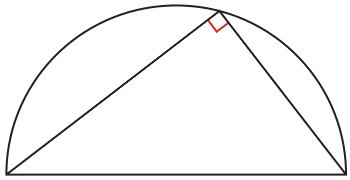
7. THE MIDPOINT THEOREM AND ITS CONVERSE

The line joining the midpoints of two sides of a triangle is parallel to the third side and is half its length.



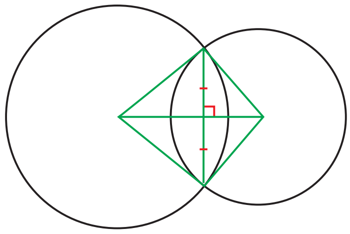
8. THE ANGLE IN A SEMI-CIRCLE

The angle subtended by any diameter of a circle at the circumference is a right angle.



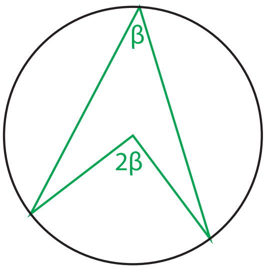
9. TWO CIRCLE THEOREM

The line joining the centres of two intersecting circles bisects the common chord at right angles.



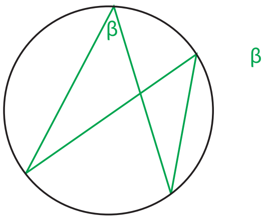
10. THE ANGLE AT THE CENTRE OF A CIRCLE

The angle subtended at the centre of a circle is twice the angle at the circle subtended by the same arc.



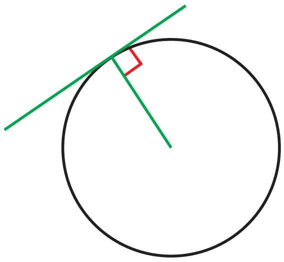
11. ANGLES IN THE SAME SEGMENT OF A CIRCLE

Angles subtended by the same arc of a circle are equal.



12. TANGENT TO A CIRCLE

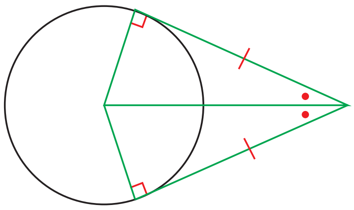
A tangent to a circle is perpendicular to the radius at the point of contact.



13. EXTERNAL POINT TO A CIRCLE

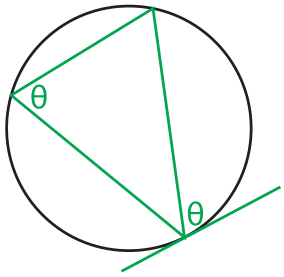
(1) The tangents drawn from an external point to a circle are equal in length.

(2) The line drawn from the centre of a circle to an external point bisects the angle between the tangents.



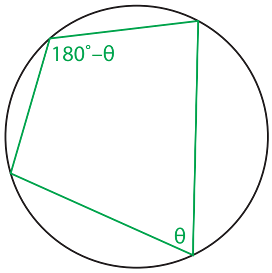
14. ANGLE BETWEEN A TANGENT AND A CHORD

The angle between a tangent and a chord at the point of contact is equal to the angle subtended by the chord in the alternate segment.



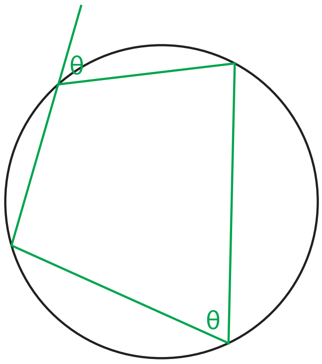
15. OPPOSITE ANGLES OF A CYCLIC QUADRILATERAL

Opposite angles of a cyclic quadrilateral are supplementary.



16. EXTERIOR ANGLE OF A CYCLIC QUADRILATERAL

The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.



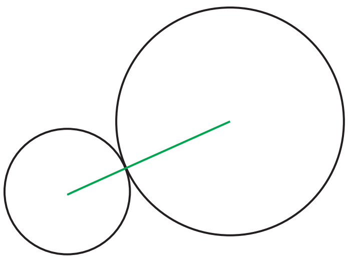
17. CONCYLIC POINTS

(1) Four points are concyclic if the line joining any two of them subtends equal angles at the other two points (on the same side of the line)

(2) A quadrilateral is cyclic if a pair of opposite angles are supplementary or if an exterior angle is equal to the interior opposite angle.

18. TOUCHING CIRCLES

If two circles touch, the point of contact lies on the line joining their centres.

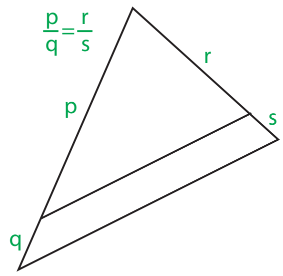


19. TRIANGLES WITH SAME ALTITUDE

The areas of triangles of the same altitude are proportional to their bases.

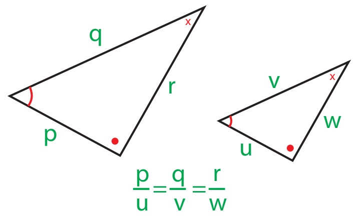
20. PARALLEL WITHIN TRIANGLE THEOREM AND ITS CONVERSE

Two sides of a triangle are cut proportionally by a straight line parallel to the third side.



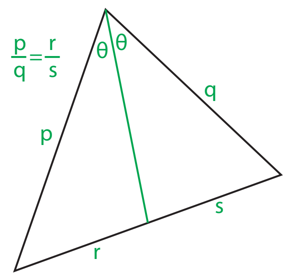
21. SIMILAR TRIANGLES THEOREM AND ITS CONVERSE

If two triangles are equiangular, their corresponding sides are proportional.



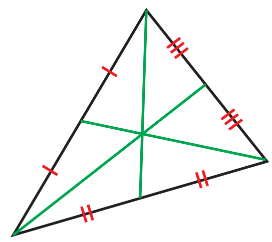
22. THE BISECTORS OF AN ANGLE OF A TRIANGLE

The bisectors of the angle of a triangle divide the opposite sides in the ratio of the sides containing them.



23. CENTROID THEOREM

The lines joining the vertices of a triangle to the mid-points of opposite sides (medians) are concurrent and trisect each other.



24. ORTHOCENTRE THEOREM

The three altitudes from the vertices to the opposite sides of a triangle are concurrent.